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D-brane Wess-Zumino terms and U-duality

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ABSTRACT: We construct gauge-invariant and U-duality covariant expressions for Wess-Zumino terms corresponding to general Dp -branes ($0 \leq p \leq D-1$) in arbitrary $3 \leq D \leq 10$ dimensions. A distinguishing feature of these Wess-Zumino terms is that they contain twice as many scalars as the $10-D$ compactified dimensions, in line with doubled geometry. We find that for $D < 10$ the charges of the higher-dimensional branes can all be expressed as products of the 0-brane charges, which include the D0-brane and the NS-NS 0-brane charges. We give the general expressions for these charges and show how they determine the non-trivial conjugacy class to which some of the higher-dimensional D-branes belong.

KEYWORDS: D-branes, String Duality

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Contents

1	Introduction	1
2	D-branes in ten dimensions	6
2.1	Type IIB	6
2.2	Type IIA	9
3	Gauge algebra in any dimension	10
4	From the A-fields to the C-fields	14
5	D-brane WZ terms	31
6	Conclusions	35

1 Introduction

Since their invention in 1995 D-branes have played a crucial role in understanding the (non-perturbative) nature of string theory [1]. D-branes occur as brane solutions of the low-energy supergravity limit of string theory [2, 3]. Their dynamics and interactions can be described by an appropriate worldvolume action. Unlike the usual brane actions the D-brane actions do not only contain worldvolume embedding scalars describing the position of the brane but also a worldvolume vector field describing the fact that Fundamental strings may end on the brane. It is well-known that the kinetic terms for the embedding scalars and worldvolume vector are given by a so-called Dirac-Born-Infeld (DBI) action. In view of this the worldvolume vector is often called the Born-Infeld (BI) vector. The coupling of the D-brane to the Ramond-Ramond (RR) gauge potentials is described by a so-called Wess-Zumino (WZ) term.¹

Using a short-hand notation the WZ terms of all Dp -branes, with $0 \leq p \leq 9$, in Type II string theory are given by [4]²

$$\mathcal{L}_{\text{WZ}}(D=10) = e^{\mathcal{F}^2} C. \quad (1.1)$$

¹A RR field is defined as a field that describes a (massless) string state created by two fermionic oscillators in the Ramond sector. Such fields couple to D-branes. Here we extend this definition to include fields that do not describe physical degrees of freedom but do couple to D-branes, e.g. domain walls and space-filling branes. These fields are related to the RR-potentials describing physical degrees of freedom via T-duality.

²We do not consider higher-order corrections involving the (target space) Riemann curvature tensor, like in [5].

Here C is defined as the formal sum³

$$C = \sum_n C_n, \quad (1.2)$$

where C_n ($1 \leq n \leq 10$) are the pull-backs of the RR n -form potentials, i.e.

$$C_{i_1 \dots i_n} = \partial_{i_1} X^{\mu_1} \dots \partial_{i_n} X^{\mu_n} C_{\mu_1 \dots \mu_n}. \quad (1.3)$$

The X^μ ($\mu = 0, 1, \dots, 10$) are the embedding worldvolume scalars and ∂_i ($i = 0, 1, \dots, p$) is the derivative with respect to the worldvolume coordinates of the Dp -brane. Furthermore, \mathcal{F}_2 is the worldvolume 2-form curvature tensor of the BI vector V_1 given by

$$\mathcal{F}_2 = dV_1 + B_2, \quad (1.4)$$

where B_2 is the Neveu-Schwarz (NS-NS) 2-form potential. Using the above notation and the Bianchi identity

$$d\mathcal{F}_2 = H_3 \equiv dB_2 \quad (1.5)$$

one can show that the WZ term is invariant under

$$\delta C = d\lambda + H_3 \wedge \lambda, \quad (1.6)$$

where λ is the formal sum of the RR gauge parameters λ_n ($0 \leq n \leq 9$):

$$\lambda = \sum_n \lambda_n. \quad (1.7)$$

Type IIA string theory has Dp -branes for all even p ($p = 0, 2, 4, 6, 8$). The D0-brane is a particle, the D2-brane a membrane etc. The D8-brane is a domain-wall (one transverse direction) which couples to the RR 9-form potential [7] that is dual to the Romans parameter m in massive IIA supergravity [8]. On the other hand, Type IIB string theory has Dp -branes for all odd p ($p = 1, 3, 5, 7, 9$). The D1-brane is the D-string etc. The D9-brane is a so-called space-filling brane that couples to the RR 10-form C_{10} of IIB supergravity. Note that the 10-form potential does not describe any physical degree of freedom and is therefore often not given in combination with the standard IIB supergravity multiplet. Nevertheless, it is needed to describe the coupling of the D9-brane to IIB supergravity and it is perfectly consistent with the IIB superalgebra to add this potential to the standard multiplet [9].

A distinguishing feature of Type IIB string theory is that it has a manifest $SL(2, \mathbb{R})$ S-duality under which the D-branes transform non-trivially. For instance, the D1-brane transforms, together with the Fundamental string F1, as a doublet of $SL(2, \mathbb{R})$. This can be seen from the fact that IIB supergravity has a doublet of 2-form potentials A_2^α ($\alpha = 1, 2$). The two components of the doublet describe the NS-NS and RR 2-forms. Similarly, the

³Note that this sum also contains a term involving the axion field C_0 which is not required by gauge-invariance of the IIB theory. It can be predicted by combining gauge invariance of the IIA theory and T-duality [6]. In this work we will first concentrate on gauge invariance and only afterwards, see end of section 5, consider the dependence of the Wess-Zumino term on the axionic scalars.

other even-form potentials of IIB supergravity transform as a singlet, doublet, triplet and quadruplet+doublet, respectively:

$$A_2^\alpha, A_4, A_6^\alpha, A_8^{\alpha\beta}, A_{10}^{\alpha\beta\gamma}, A_{10}^\alpha. \quad (1.8)$$

The 10-forms are special in the sense that they occur as a reducible representation of the duality group: a quadruplet and a doublet. It turns out that it is the quadruplet that contains the RR 10-form that couples to the D9-brane [9].

In order to write down the WZ terms for the D-branes of Type IIB string theory one needs to relate the S-duality covariant A -fields given in (1.8) to the RR-fields C occurring in the expression (1.1) of the WZ term. In order to do this in a duality-covariant way one first introduces charge vectors $\tilde{q}_\alpha, q_\alpha$ and defines [10]

$$C_2 = \tilde{q}_\alpha A_2^\alpha, \quad B_2 = q_\alpha A_2^\alpha, \quad (1.9)$$

with $\tilde{q}_\alpha q_\beta \epsilon^{\alpha\beta} \neq 0$,⁴ such that C_2 is the RR 2-form coupling to the D1-string and B_2 is the NS-NS 2-form coupling to the Fundamental string. Assuming that \tilde{q}_α and q_α transform as doublets under $SL(2, \mathbb{R})$ this fixes in a duality-covariant way which component of the 2-form doublet we define as the RR 2-form and which component as the NS-NS 2-form.

The leading term in the WZ-term of the D3-brane, i.e. (the pull-back of) the RR 4-form can be written as [10]

$$C_4 = \tilde{q}_\alpha q_\beta \left(-i\epsilon^{\alpha\beta} A_4 - \frac{1}{16} A_2^\alpha A_2^\beta \right). \quad (1.10)$$

The difference between A_4 and C_4 is that A_4 transforms under both the RR gauge transformations of C_2, C_4 and the NS-NS gauge transformations of B_2 whereas C_4 only transforms under the RR gauge transformations. The requirement that C_4 transforms as in (1.6), which is required for the gauge invariance of the WZ term (1.1), fixes the relation (1.10) between the RR-field C_4 and the duality-covariant A -fields. Extending this to the higher-rank n -forms the general WZ-term can be written in the universal form (1.1). All C -fields are related in a duality-covariant way to the A -fields, using the basic charges $\tilde{q}_\alpha, q_\alpha$. The world-volume 2-form curvature \mathcal{F}_2 is given by

$$\mathcal{F}_2 = dV_1 + B_2 = q_\alpha \mathcal{F}_2^\alpha = q_\alpha (dV_1^\alpha + A_2^\alpha). \quad (1.11)$$

Here we have introduced, together with the doublet of 2-form potentials A_2^α , a doublet of worldvolume vectors V_1^α . In this way the charges of all Type IIB Dp -branes with $p \geq 3$ can be expressed as products of the basic 1-brane charges $\tilde{q}_\alpha, q_\alpha$, corresponding to D1-branes and F1-strings, respectively [10]. The same analysis shows that

- in the case of the 10-forms, which belong to a reducible representation of $SL(2, \mathbb{R})$, it is the quadruplet, that is the highest-dimensional representation, that contains the RR field C_{10} . One cannot write down a gauge-invariant WZ term for the doublet;

⁴In [10] we took $\tilde{q}_\alpha q_\beta \epsilon^{\alpha\beta} = i$. In this paper we will leave the normalisation unfixed so that we can factor out the charges as a common factor in front of the WZ term.

dimension D	duality group G
11	1
10A	\mathbb{R}^+
10B	$\text{SL}(2)$
9	$\text{GL}(2)$
8	$\text{SL}(3) \times \text{SL}(2)$
7	$\text{SL}(5)$
6	$\text{SO}(5,5)$
5	E_6
4	E_7
3	E_8

Table 1. The U-duality groups for all maximal supergravities in dimensions $3 \leq D \leq 11$. The group is always over the real numbers and of split real form. In $D = 10$ we distinguish between IIA and IIB supergravity.

- for both the triplet of 8-forms and the quadruplet of 10-forms, not all n -forms inside the representation can be reached by a duality rotation of the RR-field. More specifically, there is one 8-form out of the triplet of 8-forms and there are two 10-forms out of the quadruplet of 10-forms that cannot be viewed as a duality transformation of the RR n -form. For these potentials one cannot write down a gauge-invariant WZ term.

It is well-known how the $\text{SL}(2, \mathbb{R})$ symmetry of IIB supergravity gets generalised to other symmetry groups for the $D < 10$ maximal supergravities [11, 12]. These symmetries have been identified as the U-dualities, containing S- and T-dualities, of superstring theory [13]. They have also been discussed in the context of worldvolume actions of extended objects in supergravity backgrounds [14]. The different duality groups for $3 \leq D \leq 11$ are given in table 1. The generic global symmetry group in D dimensions ($D \leq 9$) can be denoted (generalising what one gets in 3,4 and 5 dimensions) by E_{11-D} , where $11 - D$ is the rank of the group.

In each dimension the corresponding maximal supergravity theory contains a number of n -forms that transform in given representations of the U-duality group. These representations naturally follow by making a level decomposition of the very extended Kac-Moody algebra E_{11} [15–19]. Following the notation of [20], for each n -form we denote these representations with a lower M_n index. The forms are thus denoted by

$$A_{1,M_1}, \quad A_{2,M_2}, \quad A_{3,M_3}, \quad \dots, \quad A_{D-1,M_{D-1}}, \quad A_{D,M_D}. \quad (1.12)$$

All fields decompose into representations of the T-duality group $\text{SO}(10 - D, 10 - D)$, which is a subgroup of the U-duality group E_{11-D} . It is convenient to make this decomposition since all RR fields transform as irreducible representations of the T-duality group. A convenient way to see whether a given n -form potential is a RR field is by calculating the corresponding brane tension. For a RR field this tension should scale as $1/g_s$ in the string

frame, with g_s the string coupling constant. It turns out that in each dimensions D the RR fields of odd rank transform in the spinor representation of $SO(10-D, 10-D)$ and we denote them with $C_{2n-1,a}$, with a an $SO(10-D, 10-D)$ spinor index, while the RR fields of even rank transform in the conjugate representation and we denote them with $C_{2n,\dot{a}}$ [21]. Besides the RR-fields we need to consider the Fundamental 2-forms and 1-forms that couple to the Fundamental string and Fundamental 0-branes, i.e. wrapped Fundamental strings, respectively. These Fundamental fields have corresponding brane tensions that are independent of the string coupling constant, again in the string frame. In each dimension the Fundamental 1-forms transform in the vector representation of $SO(10-D, 10-D)$ and we denote them with $B_{1,A}$, while the Fundamental 2-form is a T-duality singlet, B_2 .

As we will discuss in detail in section 4, the equivalent of the charges $\tilde{q}_\alpha, q_\alpha$ introduced in Type IIB string theory will be charges $\tilde{q}_a^{M_1}, q_A^{M_1}$, such that

$$C_{1,a} = \tilde{q}_a^{M_1} A_{1,M_1}, \quad B_{1,A} = q_A^M A_{1,M_1} \quad (1.13)$$

define the RR and Fundamental 1-forms in a U-duality covariant way. We will see in section 4 how all the charges of the higher-dimensional Dp -branes, with $p \geq 1$, can be expressed as products of these basic 0-brane charges. Unlike in Type IIB string theory, all charges can be expressed in terms of 0-brane charges only, no 1-brane charges are involved. This has to do with the fact that, for $D < 10$, the basic gauge symmetries generating the whole gauge algebra are always the ones corresponding to the 1-form potentials only.

The aim of this paper is to construct gauge-invariant and duality-covariant expressions for general WZ terms using the ingredients introduced above. These WZ terms will describe the coupling of general Dp -branes in dimensions $3 \leq D \leq 10$ to the target space supergravity fields in a duality-covariant way. For $D = 10$ there have been attempts to construct such WZ terms. For instance, an $SL(2, \mathbb{R})$ -invariant formulation of 1-branes has been given [22, 23]. This formulation made use of the fact that in two spacetime dimensions the Born-Infeld vector is equivalent to an integration constant describing the tension of a string. Similarly, the case of 3-branes has been discussed [24]. In this case one makes use of the fact that in 4 spacetime dimensions the electric-magnetic dual of a Born-Infeld vector is again a vector. Such special properties do not occur for the branes with $p > 3$ and indeed constructing an $SL(2, \mathbb{R})$ -invariant formulation of 5-branes turns out to be problematic [25]. For $D = 10$ this gap was filled and gauge-invariant and duality-covariant expressions for all the D-branes of IIB string theory were given [10]. In this paper we extend this work to $D < 10$ dimensions.

A basic difference with $D = 10$ dimensions is that, to write down a WZ-term in $D < 10$ dimensions, we need to introduce, together with the standard worldvolume scalars describing the position of the brane in D dimensions, not only a worldvolume vector V_1 but also additional worldvolume scalars. This is due to the fact that the Fundamental string can wrap around each of the $10 - D$ compactified dimensions. Therefore, in $D < 10$ not only strings but also a number of particles can couple to the D-brane. The coupling of these particles are described by the extra worldvolume scalars. We find $2(10 - D)$ of such scalars transforming as a vector $V_{0,A}$ under the T-duality group $SO(10 - D, 10 - D)$, with

corresponding curvatures $\mathcal{F}_{1,A}$. In this paper we will show that, using these curvatures, the general WZ term can be written in the following elegant form:

$$\mathcal{L}_{\text{WZ}}(D \leq 10) = e^{\mathcal{F}_2} e^{\mathcal{F}_{1,A} \Gamma^A} C, \quad (1.14)$$

where Γ^A are the $\text{SO}(10 - D, 10 - D)$ gamma matrices and C is a formal sum of all RR-potentials. Note that $\mathcal{F}_{1,A} = 0$ in $D = 10$ since there is no T-duality in that dimension. Therefore, the expression (1.14) reduces to the usual expression (1.1) for $D = 10$.

To show that (1.14) is the correct gauge-invariant and duality-covariant WZ term for any Dp -brane in $D \leq 10$ dimensions, our strategy will be as follows. First, for the convenience of the reader, we will shortly review in section 2 the WZ terms for the D-branes of $D = 10$ Type IIA and Type IIB string theory. Next, we will present in section 3 the general gauge algebra of maximal supergravity in any dimension $3 \leq D \leq 10$. The structure of this gauge algebra follows from the underlying E_{11} algebra. We will first give a general analysis and, next, work out the formulae for each specific dimension. In section 4 we will, starting from the duality-covariant A -basis presented in section 3, derive the expressions for the RR potentials C that couple to the Dp -branes using the basic charge vectors (1.13). In particular, we will give expressions for the charge vectors of the higher-dimensional branes in terms of products of the basic charges (1.13). Like in section 3, we will first give the general analysis and then give explicit expressions for different dimensions. Next, in section 5 we will derive the main result of this paper, i.e. the gauge-invariant and duality-covariant WZ term (1.14). Finally, in section 6 we will present our conclusions and indicate a few natural extensions of this work.

2 D-branes in ten dimensions

For the convenience of the reader we shortly summarise in this section what is known about the WZ terms corresponding to the D-branes of Type IIA and Type IIB superstring theory. We first discuss the IIB case [10].

2.1 Type IIB

Our starting point is the set of $\text{SL}(2, \mathbb{R})$ -covariant IIB n -form potentials given in (1.8). Note that the 4-form field A_4 in (1.8) is self-dual and that A_2 and A_6 are each other's dual. The triplet of 8-forms is dual to the 2 scalars (axion and dilaton) of IIB supergravity. The counting works (2 scalars are dual to 3 8-forms) since the 8-forms satisfy a single constraint [9, 26, 27]. We next introduce the basic charges $q_\alpha, \tilde{q}_\alpha$ as in (1.9), where q_α and \tilde{q}_α are such that $\epsilon^{\alpha\beta} \tilde{q}_\alpha q_\beta \neq 0$. This fixes our choice of the RR potential C_2 and NS-NS potential B_2 . We are using the normalisations of [9], and convert these results in form language in the usual way:

$$A_n = \frac{1}{n!} A_{\mu_1 \dots \mu_n} dx^{\mu_1} \wedge dx^{\mu_2} \wedge \dots \wedge dx^{\mu_n} \quad . \quad (2.1)$$

Our first task is to select among all n -form fields of IIB supergravity the RR-potentials who have the defining property that they do not transform under the NS-NS gauge transformations or, equivalently, that they couple to D-branes whose tensions scale as $1/g_s$. To

see how this works, consider the gauge transformation of the 4-form potential A_4

$$\delta A_4 = d\Lambda_3 - \frac{i}{16} \epsilon_{\alpha\beta} \Lambda_1^\alpha F_3^\beta, \quad (2.2)$$

which transforms both under RR and NS-NS gauge transformations, with parameters $\tilde{q}_\alpha \Lambda^\alpha$ and $q_\alpha \Lambda^\alpha$, respectively. One may verify that there is a unique combination of A_4 and $A_2 A_2$ terms that is invariant under the NS-NS gauge transformations. This combination defines the RR 4-form potential C_4 :

$$C_4 = \tilde{q}_\alpha q_\beta \left(-i\epsilon^{\alpha\beta} A_4 - \frac{1}{16} A_2^\alpha A_2^\beta \right). \quad (2.3)$$

Applying the same procedure to the higher-form potentials we obtain the following expressions for the RR-potentials in terms of the $\text{SL}(2, \mathbb{R})$ -covariant A -fields:

$$\begin{aligned} C_6 &= \tilde{q}_\alpha q_\beta q_\gamma \left(-i\epsilon^{\alpha\beta} A_6^\gamma - \frac{4}{3} i\epsilon^{\alpha\beta} A_4 A_2^\gamma - \frac{1}{12} A_2^\alpha A_2^\beta A_2^\gamma \right), \\ C_8 &= \tilde{q}_\alpha q_\beta q_\gamma q_\delta \left(-i\epsilon^{\alpha\beta} A_8^{\gamma\delta} - \frac{i}{16} \epsilon^{\alpha\beta} A_6^\gamma A_2^\delta - \frac{i}{12} \epsilon^{\alpha\beta} A_4 A_2^\gamma A_2^\delta - \frac{1}{192} A_2^\alpha A_2^\beta A_2^\gamma A_2^\delta \right), \\ C_{10} &= \tilde{q}_\alpha q_\beta q_\gamma q_\delta q_\epsilon \left(-i\epsilon^{\alpha\beta} A_{10}^{\gamma\delta\epsilon} + \frac{i}{15} \epsilon^{\alpha\beta} A_8^{\gamma\delta} A_2^\epsilon + \frac{i}{240} \epsilon^{\alpha\beta} A_6^\gamma A_2^\delta A_2^\epsilon + \frac{i}{180} \epsilon^{\alpha\beta} A_4 A_2^\gamma A_2^\delta A_2^\epsilon \right. \\ &\quad \left. + \frac{1}{2880} A_2^\alpha A_2^\beta A_2^\gamma A_2^\delta A_2^\epsilon \right). \end{aligned} \quad (2.4)$$

These RR n -form potentials occur as the representations $\mathbf{1}_{n/2-2}$ in the decomposition of the $\text{SL}(2, \mathbb{R})$ -covariant A -fields under

$$\text{SL}(2, \mathbb{R}) \supset \mathbb{R}^+, \quad (2.5)$$

where the sub-index $n/2-2$ indicates the \mathbb{R}^+ -charge w of the representation. The complete decomposition is given in table 2. Each field can be associated with a p -brane ($p = n - 1$) if it exists, whose brane tension in string frame scales as⁵

$$g_s^\alpha, \quad \alpha = \frac{1}{2} \left(-\frac{n}{2} + w \right). \quad (2.6)$$

We distinguish between the following objects:

$$\begin{aligned} \alpha = -1 &: \quad \text{D-brane}, \\ \alpha = 0 &: \quad \text{Fundamental Object}, \\ \alpha = -2 &: \quad \text{Solitonic Object}, \\ \alpha < -2 &: \quad \text{Rest}, \end{aligned}$$

which in the table are given in the columns RR, F, S and Rest, respectively.

⁵This general formula only applies if the n -form transforms under supersymmetry to the gravitino with a non-zero coefficient that only depends on the dilaton and not the axion.

field	U repr	RR	F	S	Rest
2-form	2	$\mathbf{1}_{-1}$	$\mathbf{1}_1$		
4-form	1	$\mathbf{1}_0$			
6-form	2	$\mathbf{1}_1$		$\mathbf{1}_{-1}$	
8-form	3	$\mathbf{1}_2$		" $\mathbf{1}_0$ "	$\mathbf{1}_{-2}$
10-form	4	$\mathbf{1}_3$		" $\mathbf{1}_1$ "	" $\mathbf{1}_{-1}$ " + $\mathbf{1}_{-3}$
	2			" $\mathbf{1}_1$ "	" $\mathbf{1}_{-1}$ "

Table 2. The ten-dimensional IIB case: the 2nd column indicates the $\text{SL}(2, \mathbb{R})$ representation of the A -fields. The 3rd, 4th and 5th column indicate the Ramond-Ramond, Fundamental and Solitonic fields, respectively. The last column contains all fields with different dilaton couplings. It is not clear whether the 8-forms and 10-forms indicated by accolades couple to a brane (see the text).

We find the following expression of the charge of a Dp -brane in terms of the D1-brane and F1-brane charges:

$$\tilde{q}_{\alpha_1 \dots \alpha_{m-1}} = \left(\tilde{q}_\alpha q_\beta \epsilon^{\alpha\beta} \right) q_{\alpha_1} \dots q_{\alpha_{m-1}}, \quad p = 2m + 1. \quad (2.7)$$

Indicating with $n(q)$ and $n(\tilde{q})$ the number of q and \tilde{q} charges that occur in the expression (2.7) we have the following relations:

$$\alpha = -n(\tilde{q}), \quad w = -n(\tilde{q}) + n(q). \quad (2.8)$$

Note that there are Dp -branes for each odd p but that there is only one Fundamental string F1. The $\mathbf{1}_1$ 6-form is dual to the $\mathbf{1}_{-1}$ 2-form and represents the duality between the D5-brane and D1-brane. The $\mathbf{1}_{-1}$ 6-form is dual to the $\mathbf{1}_1$ 2-form and represents the duality between the NS 5-brane NS5B and the Fundamental F1 string.

The $\mathbf{1}_2$ 8-form couples to the D7-brane with charge

$$\mathbf{1}_2 : \quad \left(\tilde{q}_\alpha q_\beta \epsilon^{\alpha\beta} \right) q_\gamma q_\delta. \quad (2.9)$$

The other two 8-forms in the same triplet have charges:

$$\mathbf{1}_0 : \quad \left(\tilde{q}_\alpha q_\beta \epsilon^{\alpha\beta} \right) \tilde{q}_{(\gamma} q_{\delta)}, \quad \mathbf{1}_{-2} : \quad \left(\tilde{q}_\alpha q_\beta \epsilon^{\alpha\beta} \right) \tilde{q}_\gamma \tilde{q}_\delta. \quad (2.10)$$

We observe that the $\mathbf{1}_2$ D7-brane and its S-dual, the $\mathbf{1}_{-2}$ S7-brane, both have charges that are proportional to the product of two uncontracted $\text{SL}(2, \mathbb{R})$ vectors (either q_α or \tilde{q}_α) that are the same whereas the charge corresponding to the $\mathbf{1}_0$ 8-form contains the product of two *different* charges. This means that under a U-duality the D7-brane and S7-brane can be transformed into each other but that one can never rotate one of these branes into a brane corresponding to the $\mathbf{1}_0$ 8-form. In other words, the D7-brane and S7-brane belong to the same conjugacy class that forms a (non-linear) doublet embedded into the triplet. The fact that they belong to the same conjugacy class can also be deduced from the fact that, viewed

as a 2×2 matrix, both charges have zero determinant, i.e. $\det[q_\alpha q_\beta] = \det[\tilde{q}_\alpha \tilde{q}_\beta] = 0$. It is not clear whether the $\mathbf{1}_0$ 8-form couples to a brane since under supersymmetry it transforms to the gravitino with an axion-dependent coefficient, thereby violating one of the assumptions that go into the general formula (2.6). What is clear is that for such an object one can not write down a gauge-invariant WZ term as in [10] because its charge is proportional to the product of two *different* uncontracted 1-brane charge vectors.

We finally consider the 10-forms. We first consider the quadruplet of 10-forms. The $\mathbf{1}_3$ 10-form potential couples to the D9-brane and has charge

$$\mathbf{1}_3 : \quad \left(\tilde{q}_\alpha q_\beta \epsilon^{\alpha\beta} \right) q_\gamma q_\delta q_\epsilon. \quad (2.11)$$

The other three objects in the quadruplet have charges

$$\mathbf{1}_1 : \quad \left(\tilde{q}_\alpha q_\beta \epsilon^{\alpha\beta} \right) \tilde{q}_{(\gamma} q_\delta q_\epsilon), \quad \mathbf{1}_{-1} : \quad \left(\tilde{q}_\alpha q_\beta \epsilon^{\alpha\beta} \right) \tilde{q}_{(\gamma} \tilde{q}_\delta q_\epsilon), \quad \mathbf{1}_{-3} : \quad \left(\tilde{q}_\alpha q_\beta \epsilon^{\alpha\beta} \right) \tilde{q}_\gamma \tilde{q}_\delta \tilde{q}_\epsilon. \quad (2.12)$$

Applying the same reasoning as in the case of the 7-branes, we conclude that the D9-brane is in the same conjugacy class as its S-dual, the $\mathbf{1}_{-3}$ brane since these are the only two objects whose charge is proportional to the product of three $\mathrm{SL}(2, \mathbb{R})$ vectors q_α that are the same. Together, they form a non-linear doublet embedded into the quadruplet. It is not clear whether the other, $\mathbf{1}_1$ and $\mathbf{1}_{-1}$, 10-forms couple to a 9-brane since they violate the assumptions underlying (2.6). Anyway, for these two quantities it is impossible to write a gauge-invariant WZ term as in [10].

We next consider the charges associated to the doublet of ten-forms. These are

$$\mathbf{1}_1 : \quad \left(\tilde{q}_\alpha q_\beta \epsilon^{\alpha\beta} \right) \left(\tilde{q}_\gamma q_\delta \epsilon^{\gamma\delta} \right) q_\epsilon, \quad \mathbf{1}_{-1} : \quad \left(\tilde{q}_\alpha q_\beta \epsilon^{\alpha\beta} \right) \left(\tilde{q}_\gamma q_\delta \epsilon^{\gamma\delta} \right) \tilde{q}_\epsilon. \quad (2.13)$$

It turns out that using these expressions it is impossible to write down a corresponding gauge-invariant WZ term along the lines of [10].

2.2 Type IIA

Although IIA superstring theory has only an \mathbb{R}^{+-} duality symmetry the situation is similar to the IIB case. In the IIA case the charges $\tilde{q}^{(n)}$ of the higher-dimensional Dp -branes can be written as products of the D0-brane charge \tilde{q} and the Fundamental F1-string charge q as follows:

$$\tilde{q}^{(m)} = \tilde{q} q^{m/2-1/2}, \quad m = p + 1. \quad (2.14)$$

Note that, unlike in the IIB case, the basic charges correspond not only to 1-branes but also to 0-branes. The difference with the IIB case can be traced back to the fact that the IIA and IIB gauge algebras have different so-called fundamental symmetries. The fundamental symmetries are the basic gauge symmetries out of which all other symmetries can be generated by taking multiple commutators. It turns out that the fundamental symmetries of the IIB gauge algebra are the ones corresponding to the doublet of 2-form potentials, i.e. the RR and NS-NS 2-form potentials, whereas those of the IIA gauge algebra are given by the gauge symmetries corresponding to the RR 1-form and NS-NS 2-form potentials.

field	RR	F	S
1-form	$\mathbf{1}_{-1}$		
2-form		$\mathbf{1}_2$	
3-form	$\mathbf{1}_1$		
5-form	$\mathbf{1}_3$		
6-form			$\mathbf{1}_2$
7-form	$\mathbf{1}_5$		
8-form			" $\mathbf{1}_4$ "
9-form	$\mathbf{1}_7$		
10-form			" $\mathbf{1}_6$ "
			" $\mathbf{1}_6$ "

Table 3. The ten-dimensional IIA case: the 2nd, 3rd and 4th column indicates the \mathbb{R}^+ -representations of the Ramond-Ramond, Fundamental and Solitonic fields, respectively. It is not clear whether the 10-forms indicated by accolades couple to a brane.

We have collected the different n -form potentials of IIA supergravity in table 3. The corresponding brane tensions scale as,⁶

$$g_s^\alpha, \quad \alpha = -\frac{1}{2}(n - w) = -n(\tilde{q}), \quad (2.15)$$

where w is the weight under \mathbb{R}^+ , n is the rank of the form and $n(\tilde{q})$ is the number of \tilde{q} charges that occurs in the expression of the p -brane charge, with $p = n - 1$. In the second column we have indicated all RR n -form potentials with n odd. The 3d column contains the Fundamental 2-form that couples to the Fundamental string F1. The $\mathbf{1}_2$ 6-form in the 4th column couples to the solitonic 5-brane NS5A that is dual to the Fundamental F1 string. The $\mathbf{1}_4$ 8-form is the dual of the IIA dilaton. It is not clear whether it couples to a brane since under supersymmetry it does not transform to the gravitino, which was one of the assumptions going into (2.15). Finally, there are two 10-forms. For both of them it is not clear whether they couple to a brane [28].

3 Gauge algebra in any dimension

In [20] the gauge algebra of all maximal ungauged and gauged supergravities in any dimension was derived from E_{11} . Under E_{11} the fields transform as

$$\delta A = a + a \wedge A + a \wedge A \wedge A + \dots, \quad (3.1)$$

where we denote with A the form fields and with a the corresponding constant parameters. In the ungauged case, which is the one on which we will focus from now on, these

⁶This formula assumes that the n -form transforms to a gravitino with a non-zero coefficient.

transformations are promoted to gauge transformations via the identification⁷

$$a \rightarrow d\Lambda \quad , \quad (3.2)$$

and the resulting gauge algebra can be schematically written as

$$\delta A = d\Lambda + d\Lambda \wedge A + d\Lambda \wedge A \wedge A + \dots \quad . \quad (3.3)$$

Although the E_{11} algebra is non abelian, one can make, in any dimension, suitable field redefinitions, as well as field-dependent redefinitions of the gauge parameters, such that the resulting gauge algebra is abelian, that means that the gauge transformations are gauge invariant.⁸ This corresponds to writing the gauge transformations as

$$\delta A = d\Lambda + \Lambda \wedge F \quad , \quad (3.4)$$

where F 's denote the gauge-invariant field strengths. In this section we will derive the gauge algebra in this basis.

In D dimensions the global symmetry is E_{11-D} , and each n -form carries a representation of E_{11-D} that we denote with a lower M_n index [20], see eq. (1.12). Using this notation the field strength of the 1-form is

$$F_{2,M_1} = dA_{1,M_1} \quad (3.5)$$

and it is invariant under the gauge transformation

$$\delta A_{1,M_1} = d\Lambda_{0,M_1} \quad . \quad (3.6)$$

The field strength of the 2-form is

$$F_{3,M_2} = dA_{2,M_2} + f^{M_1 N_1}_{M_2} A_{1,M_1} F_{2,N_1} \quad , \quad (3.7)$$

where $f^{M_1 N_1}_{M_2}$ is an invariant tensor of E_{11-D} and an upstairs index M_n denotes the conjugate representation, which means that there is an invariant tensor $\delta^{M_n}_{N_n}$. Gauge invariance of F_{3,M_2} implies

$$\delta A_{2,M_2} = d\Lambda_{1,M_2} - f^{M_1 N_1}_{M_2} \Lambda_{0,M_1} F_{2,N_1} \quad . \quad (3.8)$$

One also gets the Bianchi identities

$$\begin{aligned} dF_{2,M_1} &= 0 \\ dF_{3,M_2} &= f^{M_1 N_1}_{M_2} F_{2,M_1} F_{2,N_1} \quad . \end{aligned} \quad (3.9)$$

The part of the invariant tensor $f^{M_1 N_1}_{M_2}$ which is antisymmetric in M_1 and N_1 can always be eliminated by means of a field redefinition $A_{2,M_2} \rightarrow A_{2,M_2} - \frac{1}{2} f^{M_1 N_1}_{M_2} A_{1,M_1} A_{1,N_1}$. We

⁷The algebraic setup underlying the promotion of the global E_{11} symmetries to local ones was constructed in [29]. The same construction applies to the case of gauged supergravities, in which the gauge algebra results from the identification $a \rightarrow d\Lambda + g\Lambda$, where g is the gauge coupling constant [20].

⁸In the context of supergravity, the existence of an abelian basis for the gauge algebra has been used in [26]. More generally, gauge algebras that do not necessarily have an abelian basis have been considered in [30].

can therefore assume that $f^{M_1 N_1}_{M_2}$ is symmetric in M_1 and N_1 . This condition naturally follows from the Jacobi identities of the E_{11} algebra.

The field strength of the 3-form is

$$F_{4,M_3} = dA_{3,M_3} + f^{M_1 M_2}_{M_3} A_{1,M_1} F_{3,M_2} + f^{M_2 M_1}_{M_3} A_{2,M_2} F_{2,M_1} \quad . \quad (3.10)$$

One can always redefine suitably the fields in such a way that the E_{11-D} invariant tensors satisfy the constraints

$$\begin{aligned} f^{M_2 M_1}_{M_3} &= 2f^{M_1 M_2}_{M_3} \quad , \\ f^{(M_1|M_2|}_{M_3} f^{N_1 P_1)}_{M_2} &= 0 \quad , \end{aligned} \quad (3.11)$$

which we therefore assume. These can also be seen as coming from E_{11} Jacobi identities. To summarise, the field strength is

$$F_{4,M_3} = dA_{3,M_3} + f^{M_1 M_2}_{M_3} [A_{1,M_1} F_{4,M_2} + 2A_{2,M_2} F_{2,M_1}] \quad , \quad (3.12)$$

and the gauge transformation of the 3-form is

$$\delta A_{3,M_3} = d\Lambda_{2,M_3} - f^{M_1 M_2}_{M_3} [\Lambda_{0,M_1} F_{3,M_2} + 2\Lambda_{1,M_2} F_{2,M_1}] \quad . \quad (3.13)$$

One also has the Bianchi identity

$$dF_{4,M_3} = 3f^{M_1 M_2}_{M_3} F_{2,M_1} F_{3,M_2} \quad . \quad (3.14)$$

For the 4-form one gets

$$F_{5,M_4} = dA_{4,M_4} + f^{M_1 M_3}_{M_4} [A_{1,M_1} F_{4,M_3} + 3A_{3,M_3} F_{2,M_1}] + f^{M_2 N_2}_{M_4} A_{2,M_2} F_{3,N_2} \quad (3.15)$$

and

$$\delta A_{4,M_4} = d\Lambda_{3,M_4} - f^{M_1 M_3}_{M_4} [\Lambda_{0,M_1} F_{4,M_3} + 3\Lambda_{2,M_3} F_{2,M_1}] - f^{M_2 N_2}_{M_4} \Lambda_{1,M_2} F_{3,N_2} \quad , \quad (3.16)$$

with the constraints

$$f^{M_2 N_2}_{M_4} f^{M_1 N_1}_{N_2} = 6f^{(M_1|M_3|}_{M_4} f^{N_1)}_{M_2} f^{N_2)}_{M_3} \quad (3.17)$$

and the constraint that $f^{M_2 N_2}_{M_4}$ is antisymmetric in M_2 and N_2 . This analysis can easily be extended to forms of higher rank.

For the convenience of the reader we write the explicit form of the various invariant tensors in dimension D from 9 to 4, for each dimension separately. This reproduces the results of [20] in the abelian basis. The notations we use here are taken from that paper. The results are summarised in table 4.

D=9. In nine dimensions the global symmetry is $GL(2, \mathbb{R}) = SL(2, \mathbb{R}) \times \mathbb{R}^+$ and the fields are

1-form	$\mathbf{2}_0 \oplus \mathbf{1}_1$	$A_{1\alpha}, A_1$
2-form	$\mathbf{2}_1$	$A_{2,\alpha}$
3-form	$\mathbf{1}_1$	A_3
4-form	$\mathbf{1}_2$	A_4 ,

where $\alpha = 1, 2$ is an $SL(2, \mathbb{R})$ doublet index and the sub-index indicates the \mathbb{R}^+ charge. The only non-trivial invariant tensor is the epsilon symbol of $SL(2, \mathbb{R})$, and from table 4 one can see that the conditions (3.11) and (3.17) are satisfied.

D	A_{1,M_1}	A_{2,M_2}	A_{3,M_3}	A_{4,M_4}	$f^{M_1 N_1}{}_{M_2}$	$f^{M_1 M_2}{}_{M_3}$	$f^{M_1 M_3}{}_{M_4}$	$f^{M_2 N_2}{}_{M_4}$
9	$A_{1,\alpha} A_1$	$A_{2,\alpha}$	A_3	A_4	δ_β^α	$\epsilon^{\alpha\beta}$	$-\frac{1}{3}$	$\epsilon^{\alpha\beta}$
8	$A_{1,M\alpha}$	A_2^M	$A_{3,\alpha}$	$A_{4,M}$	$\epsilon^{MNP} \epsilon^{\alpha\beta}$	$\delta_N^M \delta_\beta^\alpha$	$\frac{1}{12} \delta_N^M \epsilon^{\alpha\beta}$	ϵ_{MNP}
7	$A_{1,MN}$	A_2^M	$A_{3,M}$	A_4^{MN}	ϵ^{MNPQR}	$\delta_P^{[M} \delta_Q^{N]}$	$-\frac{1}{3} \epsilon^{MNPQR}$	$\delta_M^{[P} \delta_N^{Q]}$
6	$A_{1,\dot{\alpha}}$	$A_{2,M}$	$A_{3,\alpha}$	$A_{4,MN}$	$(C\Gamma_M)^{\dot{\alpha}\beta}$	$(\Gamma^M)_\alpha{}^{\dot{\alpha}}$	$-\frac{1}{12} (C\Gamma_{MN})^{\dot{\alpha}\alpha}$	$\delta_M^{[P} \delta_N^{Q]}$
5	$A_{1,M}$	A_2^M	$A_{3,\alpha}$	A_4^{MN}	d^{MNP}	$D_{\alpha,M}{}^N$	$S^{\alpha M, NP}$	$\delta_M^{[P} \delta_N^{Q]}$
4	$A_{1,M}$	$A_{2,\alpha}$	$A_{3,A}$	$A_{4,\alpha\beta}$	D_α^{MN}	$S_A^{M\alpha}$	$C_{\alpha\beta}^{MA}$	$\delta_\gamma^{[\alpha} \delta_\delta^{\beta]}$

Table 4. The E_{11-D} invariant tensors associated to all the fields up to the 4-form corresponding to dimensions D from 9 to 4. The definitions of the invariant tensors and the relations between them are taken from [20].

D=8. In eight dimensions the global symmetry is $SL(3, \mathbb{R}) \times SL(2, \mathbb{R})$ and the fields are ($M = 1, 2, 3; \alpha = 1, 2$)

$$\begin{array}{lll}
 \text{1-form} & (\bar{\mathbf{3}}, \mathbf{2}) & A_{1,M\alpha} \\
 \text{2-form} & (\mathbf{3}, \mathbf{1}) & A_2^M \\
 \text{3-form} & (\mathbf{1}, \mathbf{2}) & A_{3,\alpha} \\
 \text{4-form} & (\bar{\mathbf{3}}, \mathbf{1}) & A_{4,M} \ .
 \end{array}$$

Given the invariant tensors of table 4, the conditions (3.11) and (3.17) are identically satisfied.

D=7. In seven dimensions the symmetry is $SL(5, \mathbb{R})$, while the representations of the fields up to the 5-form and the corresponding notations are ($M = 1, \dots, 5$)

$$\begin{array}{lll}
 \text{1-form} & \bar{\mathbf{10}} & A_{1,MN} \\
 \text{2-form} & \mathbf{5} & A_2^M \\
 \text{3-form} & \bar{\mathbf{5}} & A_{3,M} \\
 \text{4-form} & \mathbf{10} & A_4^{MN} \ .
 \end{array}$$

Given the invariant tensors of table 4, one may verify that the conditions (3.11) and (3.17) are satisfied.

D=6. The global symmetry of the six-dimensional theory is $SO(5, 5)$. The representations of the fields up to the 5-form, and their corresponding notations, are ($\alpha, \dot{\alpha} = 1, \dots, 16; M = 1, \dots, 10$)

$$\begin{array}{lll}
 \text{1-form} & \mathbf{16} & A_{1,\dot{\alpha}} \\
 \text{2-form} & \mathbf{10} & A_{2,M} \\
 \text{3-form} & \bar{\mathbf{16}} & A_{3,\alpha} \\
 \text{4-form} & \mathbf{45} & A_{4,MN} \ .
 \end{array}$$

The invariant tensors are 10-dimensional Gamma matrices, and the conditions (3.11) and (3.17) are Fierz identities.

D=5. In five dimensions the global symmetry is E_6 . The representations of the fields up to the 5-form, and their corresponding notations, are ($M = 1, \dots, 27; \alpha = 1, \dots, 78$)

1-form	27	$A_{1,M}$
2-form	$\overline{\mathbf{27}}$	A_2^M
3-form	78	$A_{3,\alpha}$
4-form	351	A_4^{MN} .

It should be noted that we are considering the split, or real form of E_6 , therefore the representations are real. With $\overline{\mathbf{27}}$ we simply mean the representation that is dual to the **27**. This will be the case in all dimensions. The conditions (3.11) and (3.17) are satisfied by the invariant tensors in table 4.

D=4. Finally, in four dimensions the global symmetry is E_7 . The representations of the forms up to the 5-form and the corresponding notations are ($M = 1, \dots, 56; \alpha = 1, \dots, 133; A = 1, \dots, 912$)

1-form	56	$A_{1,M}$
2-form	133	$A_{2,\alpha}$
3-form	912	$A_{3,A}$
4-form	8645 + 133	$A_{4,\alpha\beta}$.

The index A with which we denote the **912** representation should not be confused with the A vector index of the T-duality group in any dimension. The reducible representation to which the 4-forms belong is the antisymmetric product of two adjoint (i.e. **133**) indices. Therefore the $\alpha\beta$ indices of the 4-form are meant to be antisymmetrised. Note that in all dimensions the representation of the 4-form is the antisymmetrised product of two 2-form representations.

Given the invariant tensors listed in table 4, eq. (3.11) becomes

$$S_A^{\alpha(M} D_\alpha^{NP)} = 0 \quad , \quad (3.18)$$

while eq. (3.17) is

$$\delta_{[\gamma}^\alpha D_{\delta]}^{MN} = 6C_{\gamma\delta}^{(M|A|} S_A^{N)\alpha} \quad . \quad (3.19)$$

These two equations are both satisfied, see [20].

4 From the A -fields to the C -fields

In this section we wish to generalise the IIB construction of [10], thus determining in any dimension the RR C -fields starting from the duality-covariant A -basis of the previous section. Schematically, we expect the C fields to transform under the RR gauge transformations as

$$\delta C \sim d\lambda + H \wedge \lambda \quad , \quad (4.1)$$

where λ are the gauge parameters of the RR fields and H are the field strengths of the Fundamental fields, that is the fields corresponding to the branes whose tension does not

depend on the dilaton in the string frame. In ten dimensions, H is a 3-form. By dimensional reduction, we expect H in $D < 10$ dimensions to be either a two-form or a three-form. That is why we have not yet indicated the rank of the forms in (4.1). The precise form of this equation will be given below, see eq (4.14).

The T-duality subgroup of the U-duality group E_{11-D} in D dimensions is $SO(10 - D, 10 - D)$. It is known that the RR fields transform in the spinorial representations of the T-duality group [21]. This is consistent with the fact that, upon reduction over a single dimension, each Dp -brane gives rise to two branes: a Dp -brane (dimensional reduction in the transverse direction) and a $D(p - 1)$ -brane (dimensional reduction in the worldvolume direction). Therefore, the total number of D-branes doubles when going one down in the dimension D . This is precisely what happens with the dimension d_{spinor} of a spinor representation of the T-duality group $SO(10 - D, 10 - D)$ which is given by $d_{\text{spinor}} = 2^{9-D}$. It turns out that, more precisely, the forms of odd rank transform as spinors of a given chirality, while the forms of even rank transform as spinors of the opposite chirality. As we will see in this section, this is completely general and applies to RR $(D - 1)$ - and D -forms as well.

We now decompose in any dimension each duality-covariant form, belonging to a given E_{11-D} representation denoted by M_n , in terms of representations of $SO(10 - D, 10 - D)$. More precisely, we have

$$E_{11-D} \supset SO(10 - D, 10 - D) \times \mathbb{R}^+ \quad (4.2)$$

in all cases $D < 10$. In $D = 4$ and $D = 3$ dimensions there are extra symmetry enhancements, such that in $D = 4$ the decomposition is

$$E_7 \supset SO(6, 6) \times SL(2, \mathbb{R}) \quad , \quad (4.3)$$

and in $D = 3$ one has

$$E_8 \supset SO(8, 8) \quad . \quad (4.4)$$

From the point of view of this paper these symmetry enhancements are not practical since they combine the RR-fields together with n -forms that couple to other kind of branes in one multiplet. We will therefore consider the further decompositions

$$SL(2, \mathbb{R}) \supset \mathbb{R}^+ \quad (4.5)$$

for $D = 4$ and

$$SO(8, 8) \supset SO(7, 7) \times \mathbb{R}^+ \quad (4.6)$$

for $D = 3$. In the second part of this section we will discuss the above decompositions for each dimension separately. Here, we first anticipate the main outcome of this analysis.

Using the fact that eq. (4.1) relates a RR n -form to the gauge parameter of a RR $(n - 2)$ -form, it follows that considering the decomposition (4.2), the \mathbb{R}^{+-} charges satisfy the relations

$$w_{\text{RR}}(n) = w_{\text{RR}}(n - 1) + w_{\text{F}}(1) \quad , \quad w_{\text{RR}}(n) = w_{\text{RR}}(n - 2) + w_{\text{F}}(2) \quad , \quad (4.7)$$

field	RR	F
A_{1,M_1}	$C_{1,a}$	$B_{1,A}$
A_{2,M_2}	$C_{2,\dot{a}}$	B_2
$A_{2n-1,M_{2n-1}}$	$C_{2n-1,a}$	
$A_{2n,M_{2n}}$	$C_{2n,\dot{a}}$	

Table 5. The RR and Fundamental fields in any dimension. In the last two lines n is meant to be greater than 1.

where we denote with $w_{\text{RR}}(n)$ the charge of the RR n -form and with $w_{\text{F}}(1)$ and $w_{\text{F}}(2)$ the charge of the Fundamental 1-forms and 2-forms. In tables 6–11 the decomposition of all the forms with respect to T-duality is performed in any dimension $4 \leq D \leq 9$. From an analysis of the \mathbb{R}^+ -charges in the tables, it is straightforward to see that the condition (4.7) on the charges has only one solution, and, moreover, it is the same solution in any dimension. That solution is:

- All RR-forms, including the $(D-1)$ -forms and the D -forms, belong to the spinor representations of the T-duality group. The RR-forms of odd rank transform as spinors of a given chirality, and the RR-forms of even rank transform as spinors of the opposite chirality.
- In the cases in which the representation of the U-duality group is reducible, as is always the case for D -forms, and also for $(D-1)$ -forms in dimension higher than six and for $(D-2)$ -forms in dimension higher than seven, the RR-forms are always inside the highest-dimensional representation.
- The Fundamental 1-forms always belong to the vector representation of the T-duality group. We denote these fields by $B_{1,A}$, with A a vector index of $\text{SO}(10-D, 10-D)$. Half of these represent the wrapped Fundamental strings, while the other half correspond to reduced pp-waves which, in $D=10$, are T-dual to the Fundamental string.
- The Fundamental 2-form always transforms as a singlet under T-duality. This is not surprising because this is the form associated to the Fundamental string.

In the four-dimensional case, in which the decomposition of eq. (4.3) occurs, the situation is more subtle due to the further decomposition (4.5) but the final result is exactly the same, as we will show in the last subsection of this section.

The general result is summarised in table 5. A straightforward way of reaching this conclusion for the lower rank forms is to consider explicitly the dimensional reduction of IIA and IIB supergravity. The same dimensional reduction also gives the right answer for the higher rank RR-forms that do not describe physical degrees of freedom. It turns out that all RR-forms in $D < 10$ dimensions arise from dimensional reduction of the IIA

or IIB RR-forms. This is not the case for the forms that we have collected in the ‘Rest’ columns of the different tables which have dilaton couplings different from the RR and Fundamental fields. Their higher-dimensional origin resides in the mixed representations predicted by E₁₁. These ‘Rest’ fields are a minority in $D = 10$ dimensions but become a majority in lower dimensions. Besides Solitonic objects they might describe other exotic objects in string theory, with unconventional dilaton couplings, like $1/g_s^3, 1/g_s^4$, etc. The precise meaning of these objects, if they exist et all, is not understood.

Given these general results, we now proceed with analysing how the C -basis is defined in terms of the duality-covariant A -basis. As we have already stressed and as table 5 shows, the RR-fields transform in the fermionic representations of $\text{SO}(10 - D, 10 - D)$ and we denote them by

$$C_{2n-1,a} \quad , \quad C_{2n,\dot{a}} \quad (4.8)$$

denoting with a and \dot{a} the 2^{9-D} -dimensional spinor representations of $\text{SO}(10 - D, 10 - D)$. It is useful to list the conventions for the $\text{SO}(10 - D, 10 - D)$ Gamma matrices that we are using. In particular, we are using a Weyl basis, so that the Gamma matrices have the form

$$\Gamma_A = \begin{pmatrix} 0 & (\Gamma_A)_a{}^{\dot{b}} \\ (\Gamma_A)_{\dot{a}}{}^b & 0 \end{pmatrix} \quad , \quad (4.9)$$

where $a, \dot{a} = 1, \dots, 2^{9-D}$. They satisfy the Clifford algebra

$$\{\Gamma_A, \Gamma_B\} = 2\eta_{AB} \quad (4.10)$$

where η_{AB} is the Minkowski metric.

Because of \mathbb{R}^{+-} charge conservation, we expect the NS-NS field strengths of $B_{1,A}$ and B_2 to be

$$\begin{aligned} H_{2,A} &= dB_{1,A} \\ H_3 &= dB_2 + B_{1,A}H_{2,B}\eta^{AB} \quad , \end{aligned} \quad (4.11)$$

whose gauge invariance fixes the gauge transformations of the fields to be

$$\begin{aligned} \delta B_{1,A} &= d\Sigma_{0,A} \\ \delta B_2 &= d\Sigma_1 - \Sigma_{0,A}H_{2,B}\eta^{AB} \quad . \end{aligned} \quad (4.12)$$

We write down the field strengths and gauge transformations of all the RR fields in the compact form

$$G = dC + H_3C + H_{2,A}\Gamma^A C \quad (4.13)$$

and

$$\delta C = d\lambda + H_3\lambda - H_{2,A}\Gamma^A\lambda \quad . \quad (4.14)$$

Here we denote with C the sum of all the RR forms, in the spinor representation of $\text{SO}(10 - D, 10 - D)$, where each odd form in the sum is projected on one chirality and

each even form on the opposite chirality, and the opposite projection occurs for the gauge parameters λ . In components the two equations read

$$\begin{aligned} G_{2n,a} &= dC_{2n-1,a} + H_3 C_{2n-3,a} + H_{2,A} (\Gamma^A)_a{}^{\dot{b}} C_{2n-2,\dot{b}} \quad , \\ G_{2n+1,\dot{a}} &= dC_{2n,\dot{a}} + H_3 C_{2n-2,\dot{a}} + H_{2,A} (\Gamma^A)_{\dot{a}}{}^b C_{2n-1,b} \quad , \end{aligned} \quad (4.15)$$

and

$$\begin{aligned} \delta C_{2n-1,a} &= d\lambda_{2n-2,a} + H_3 \lambda_{2n-4,a} - H_{2,A} (\Gamma^A)_a{}^{\dot{b}} \lambda_{2n-3,\dot{b}} \quad , \\ \delta C_{2n,\dot{a}} &= d\lambda_{2n-1,\dot{a}} + H_3 \lambda_{2n-3,\dot{a}} - H_{2,A} (\Gamma^A)_{\dot{a}}{}^b \lambda_{2n-2,b} \quad . \end{aligned} \quad (4.16)$$

We now introduce the charges $\tilde{q}_a^{M_1}$ and $q_A^{M_1}$ that project the 1-forms on the RR and Fundamental 1-forms respectively. That is

$$C_{1,a} = \tilde{q}_a^{M_1} A_{1,M_1} \quad , \quad B_{1,A} = q_A^{M_1} A_{1,M_1} \quad . \quad (4.17)$$

Up to field redefinitions, the most general expression for the RR and Fundamental 2-forms is

$$\begin{aligned} B_2 &= q^{M_2} A_{2,M_2} \\ C_{2,\dot{a}} &= \tilde{q}_{\dot{a}}^{M_2} A_{2,M_2} + a (\Gamma^A)_{\dot{a}}{}^b \tilde{q}_b^{M_1} q_A^{N_1} A_{1,M_1} A_{1,N_1} \quad , \end{aligned} \quad (4.18)$$

where the parameter a will now be determined by consistency. Indeed, varying both expressions according to eqs. (3.8), (4.12) and (4.14), and using the fact that $f^{M_1 N_1}_{M_2}$ is symmetric in M_1 and N_1 , one obtains $a = \frac{1}{2}$. Furthermore, consistency implies the constraints

$$\tilde{q}_{\dot{a}}^{M_2} f^{M_1 N_1}_{M_2} = (\Gamma^A)_{\dot{a}}{}^b \tilde{q}_b^{(M_1} q_A^{N_1)} \quad (4.19)$$

and

$$q^{M_2} f^{M_1 N_1}_{M_2} = q_A^{M_1} q_B^{N_1} \eta^{AB} \quad . \quad (4.20)$$

These relations can be inverted using the invariant tensor $\tilde{f}_{M_1 N_1}^{M_2}$, where in general $\tilde{f}_{M_m N_n}^{P_{m+n}}$ is such that

$$f^{M_m N_n}_{P_{m+n}} \tilde{f}_{M_m N_n}^{Q_{m+n}} = \delta_{P_{m+n}}^{Q_{m+n}} \quad , \quad (4.21)$$

and thus they determine the charges of the 2-forms entirely in terms of the charges of the 1-forms:

$$\begin{aligned} \tilde{q}_{\dot{a}}^{M_2} &= (\Gamma^A)_{\dot{a}}{}^b \tilde{q}_b^{M_1} q_A^{N_1} \tilde{f}_{M_1 N_1}^{M_2} \quad , \\ q^{M_2} &= q_A^{M_1} q_B^{N_1} \eta^{AB} \tilde{f}_{M_1 N_1}^{M_2} \quad . \end{aligned} \quad (4.22)$$

To summarise, the expressions for the gauge fields and the gauge parameters are

$$\begin{aligned} B_2 &= q_A^{M_1} q_B^{N_1} \eta^{AB} \tilde{f}_{M_1 N_1}^{M_2} A_{2,M_2} \quad , \\ C_{2,\dot{a}} &= (\Gamma^A)_{\dot{a}}{}^b \tilde{q}_b^{M_1} q_A^{N_1} \left(\tilde{f}_{M_1 N_1}^{M_2} A_{2,M_2} + \frac{1}{2} A_{1,M_1} A_{1,N_1} \right) \quad , \end{aligned} \quad (4.23)$$

and

$$\begin{aligned}\Sigma_1 &= q^{M_2} \Lambda_{1,M_2}, \\ \lambda_{1,\dot{a}} &= \tilde{q}_{\dot{a}}^{M_2} \Lambda_{1,M_2} + (\Gamma^A)_{\dot{a}}{}^b \tilde{q}_{\dot{b}}^{[M_1} q_A^{N_1]} \Lambda_{0,M_1} A_{1,N_1} \quad .\end{aligned}\quad (4.24)$$

We now move to the 3-forms. Before determining the actual expression for the RR 3-form in terms of the A -fields, we first analyse the corresponding charge. In principle, the charge $q_a^{M_3}$ can be written as either

$$(\Gamma^A)_a{}^b \tilde{q}_{\dot{b}}^{M_2} q_A^{M_1} \tilde{f}_{M_1 M_2}{}^{M_3} \quad (4.25)$$

or

$$\tilde{q}_a^{M_1} q^{M_2} \tilde{f}_{M_1 M_2}{}^{M_3} \quad . \quad (4.26)$$

Substituting the relations (4.22) and using the condition

$$\tilde{f}_{(M_1|M_2|}{}^{M_3} \tilde{f}_{N_1 P_1)}{}^{M_2} = 0 \quad , \quad (4.27)$$

which is the conjugate of the second of the constraints of eq. (3.11), one can see that the two expressions for $q_a^{M_3}$ are actually proportional, and therefore one can write

$$\tilde{q}_a^{M_3} \propto \tilde{q}_a^{M_1} q_A^{N_1} q_B^{P_1} \eta^{AB} \tilde{f}_{N_1 P_1}{}^{M_2} \tilde{f}_{M_1 M_2}{}^{M_3} \quad . \quad (4.28)$$

We are now ready to determine the relation between the RR 3-forms and the A -fields, which also determines the coefficient in (4.28). The procedure is completely general: we know the variation of the C field, that is eq. (4.14), and we know the transformation of the 3-form A field, that is eq. (3.13). We then compare the two expressions solving for C in terms of A . The final result is

$$\begin{aligned}C_{3,a} &= \tilde{q}_b^{M_1} q_A^{N_1} q_B^{P_1} \left[-\frac{1}{2} \delta_a^b \eta^{AB} \tilde{f}_{N_1 P_1}{}^{M_2} \tilde{f}_{M_1 M_2}{}^{M_3} A_{3,M_3} - \frac{1}{3} (\Gamma^A \Gamma^B)_a{}^b \tilde{f}_{M_1 P_1}{}^{M_2} A_{2,M_2} A_{1,N_1} \right. \\ &\quad \left. - \frac{2}{3} \delta_a^b \eta^{AB} \tilde{f}_{N_1 P_1}{}^M A_{1,M_1} A_{2,M_2} + \frac{1}{6} (\Gamma^{AB})_a{}^b A_{1,M_1} A_{1,N_1} A_{1,P_1} \right] .\end{aligned}\quad (4.29)$$

Proceeding this way one can determine the relation for all the RR C -fields in terms of the duality-covariant A -fields. These results are general and apply to any dimension. We give the expression for the charge of any D-brane in terms of the charges $\tilde{q}_a^{M_1}$ and $q_A^{M_1}$. For instance, for the 4-form one gets

$$\tilde{q}_a^{M_4} \propto (\Gamma^A)_{\dot{a}}{}^b \eta^{BC} \tilde{q}_{\dot{b}}^{M_1} q_A^{N_1} q_B^{P_1} q_C^{Q_1} \tilde{f}_{P_1 Q_1}{}^{M_2} \tilde{f}_{N_1 M_2}{}^{M_3} \tilde{f}_{M_1 M_3}{}^{M_4} \quad . \quad (4.30)$$

This expression is unique, in the sense that there is no other independent way of contracting one $\tilde{q}_a^{M_1}$ with three $q_A^{M_1}$'s to get an object with the right indices. The reason is, like in the discussion of the 3-forms above, the constraints that the \tilde{f} generalised structure constants satisfy for consistency of the gauge algebra. In this case the relevant constraint is the conjugate of eq. (3.17). One can show that this is true in all cases, and the general expression for the charge is

$$\tilde{q}_a^{M_{2n+1}} \propto \tilde{q}_a^{M_1} q_{A^{(1)}}^{N_1^{(1)}} q_{B^{(1)}}^{P_1^{(1)}} \eta^{A^{(1)} B^{(1)}} \cdots q_{A^{(n)}}^{N_1^{(n)}} q_{B^{(n)}}^{P_1^{(n)}} \eta^{A^{(n)} B^{(n)}} \tilde{f}_{M_1 N_1^{(1)}}{}^{M_2} \cdots \tilde{f}_{P_1^{(n)} M_{2n}}{}^{M_{2n+1}} \quad (4.31)$$

for odd forms and

$$\tilde{q}_{\dot{a}}^{M_{2n+2}} \propto (\Gamma^A)_{\dot{a}}{}^b \tilde{q}_b^{M_1} q_A^{N_1} q_{A^{(1)}}^{N_1^{(1)}} q_{B^{(1)}}^{P_1^{(1)}} \eta^{A^{(1)}B^{(1)}} \dots q_{A^{(n)}}^{N_1^{(n)}} q_{B^{(n)}}^{P_1^{(n)}} \eta^{A^{(n)}B^{(n)}} \tilde{f}_{M_1 N_1}{}^{M_2} \dots \tilde{f}_{P_1^{(1)} M_{2n+1}}{}^{M_{2n+2}} \quad (4.32)$$

for even forms.

Clearly one can in general construct charges with $n(\tilde{q}) \neq 1$, which do not correspond to D-branes. It turns out that there is a simple formula for the dilaton scaling of the tension of the brane to which the n -forms couple. It is given in terms of the number $n(\tilde{q})$ of basic \tilde{q} charges that one uses in the above expressions. All tensions scale as

$$g_s^\alpha, \quad \alpha = -n(\tilde{q}). \quad (4.33)$$

This formula assumes that the corresponding n -form transforms under supersymmetry to the gravitino with a non-zero coefficient that only depends on the dilaton scalar.

The structure for the charges we found above simplifies if we consider only those charges that are generated by 2-form basic charges. This corresponds to considering the gauge algebra of the even-form fields whose gauge transformations are generated by the 2-form gauge transformation, that is considering only the structure constants $f^{M_m N_n}{}_{P_{m+n}}$ where m and n are even. The 2-brane charges are in fact given in terms of the 1-form charges in (4.22), but now we consider them as basic charges and we indicate them schematically with Q in order to distinguish them from the original 1-form charges q . We will make use of this observation when we discuss the issue of conjugacy classes for the $D = 8$ case below. The simplification is that when one projects on this sector, in the original expression for the higher-form charges in terms of the 1-form basic charges q all 1-form basic charges pair up to 2-form basic charges $Q \sim qq$. More specifically, assuming that the 2-form sector only contains RR and Fundamental 2-forms, which is true for $D > 6$,⁹ the basic charges are given by

$$\tilde{Q}_{\dot{a}}^{M_2}, \quad Q^{M_2}. \quad (4.34)$$

These basic charges define the RR and Fundamental 2-forms as

$$C_{2,\dot{a}} = \tilde{Q}_{\dot{a}}^{M_2} A_{2,M_2}, \quad B_2 = Q^{M_2} A_{2,M_2}. \quad (4.35)$$

We next consider the gauge algebra restricted to this sector and, for simplicity, we consider only the gauge transformations of the 2-forms and 4-forms:

$$\begin{aligned} \delta A_{M_2} &= d\Lambda_{M_2} \\ \delta A_{M_4} &= \delta\Lambda_{M_4} - f^{M_2 N_2}{}_{M_4} \Lambda_{M_2} F_{N_2}, \end{aligned} \quad (4.36)$$

where $f^{M_2 N_2}{}_{M_4}$ is antisymmetric in $M_2 N_2$. The remarkable thing about this subsector is that in all cases, that is in any dimensions, the invariant tensor $f^{M_2 N_2}{}_{M_4}$ does not put any constraint, that is the representation M_4 is in all cases precisely the antisymmetric product

⁹For $D \leq 6$ one should include the charge of the solitonic 1-brane as well, but this does not affect the analysis of the RR sector.

of two M_2 representations [20]. The consistency of the gauge transformations requires the $C_{4,\dot{a}}$ RR-field to be expressed in terms of the U-duality-covariant A -fields as

$$C_{4,\dot{a}} = \tilde{Q}_{\dot{a}}^{M_2} Q^{N_2} \left(\tilde{f}_{M_2 N_2}^{M_4} A_{4,M_4} - \frac{1}{2} A_{2,M_2} A_{2,N_2} \right) , \quad (4.37)$$

where the charge $Q_{\dot{a}}^{M_4}$ is

$$\tilde{Q}_{\dot{a}}^{M_4} = \tilde{Q}_{\dot{a}}^{M_2} Q^{N_2} \tilde{f}_{M_2 N_2}^{M_4} . \quad (4.38)$$

The difference with respect to the full case is that no invariant tensor of $SO(10-D, 10-D)$ takes part in the expression (4.38). This is because the Fundamental 2-form charge is a singlet. The only reason why this is consistent is that neither does $\tilde{f}_{M_2 N_2}^{M_4}$ pose any constraint on the representations. We therefore arrive at the conclusion that there is a collaboration between the universal unconstrained structure of the gauge algebra when restricted to the forms generated by the even-form basic charges only, and the fact that there are fundamental strings in the theory that are singlets under T-duality. In other words, the gauge algebra knows about strings!

Below we show how things work out for each dimension $4 \leq D \leq 9$ separately, starting with the highest dimension.

D=9. The U-duality symmetry of maximal supergravity in nine dimensions is $SL(2, \mathbb{R}) \times \mathbb{R}^+$. One can consider the nine-dimensional theory (as well as any lower dimensional one) as coming from dimensional reduction of either the IIA or the IIB theory. It is instructive to review how the branes in the IIA theory can be seen from 11 dimensions, which is summarised as

$$\begin{aligned} g_{\mu}^{\sharp} &\rightarrow \text{D0} & A_{\mu\nu\sharp} &\rightarrow \text{F1} \\ A_{\mu\nu\rho} &\rightarrow \text{D2} & A_{\mu_1 \dots \mu_5 \sharp} &\rightarrow \text{D4} \\ A_{\mu_1 \dots \mu_6} &\rightarrow \text{NS5A} & & \end{aligned} \quad (4.39)$$

Here we denote with \sharp the compact 11th coordinate, and the 6-form A_6 is the magnetic dual of the 3-form A_3 in eleven dimensions.

Reducing to nine dimensions results in the fields collecting in $SL(2, \mathbb{R})$ multiplets, as it is obvious from the 11-dimensional or IIB origin of the theory. We analyse the fields from the IIA/11-dimensional viewpoint.¹⁰ Denoting with 9 the compact 10th coordinate, the 1-forms, and the corresponding 0-branes, are

$$(g_{\mu}^{\sharp}, g_{\mu}^9) \rightarrow (\text{D0}, \text{F0}) \quad A_{\mu 9 \sharp} \rightarrow \text{F0} , \quad (4.40)$$

while the 2-forms, and the corresponding 1-branes, are

$$(A_{\mu\nu 9}, A_{\mu\nu \sharp}) \rightarrow (\text{D1}, \text{F1}) , \quad (4.41)$$

¹⁰ From the IIB point of view one obtains the same results, only the ten-dimensional origin is different. What is a reduced pp-wave from the IIA point of view becomes a wrapped Fundamental string from the IIB point of view. Nothing changes for RR fields.

field	U repr	RR	F	Rest
1-form	$\mathbf{2}_0$	$(-1, 0)$	$(1, 0)$	
	$\mathbf{1}_1$		$(0, 1)$	
2-form	$\mathbf{2}_1$	$(-1, 1)$	$(1, 1)$	
3-form	$\mathbf{1}_1$	$(0, 1)$		
4-form	$\mathbf{1}_2$	$(0, 2)$		
5-form	$\mathbf{2}_2$	$(1, 2)$		$(-1, 2)$
6-form	$\mathbf{2}_3$	$(1, 3)$		$(-1, 3)$
	$\mathbf{1}_2$			$(0, 2)$
7-form	$\mathbf{3}_3$	$(2, 3)$		$(0, 3) + (-2, 3)$
	$\mathbf{1}_3$			$(0, 3)$
8-form	$\mathbf{3}_4$	$(2, 4)$		$(0, 4) + (-2, 4)$
	$\mathbf{2}_3$			$(1, 3) + (-1, 3)$
9-form	$\mathbf{4}_4$	$(3, 4)$		$(1, 4) + (-1, 4) + (-3, 4)$
	$2 \times \mathbf{2}_4$			$2 \times [(1, 4) + (-1, 4)]$

Table 6. The decomposition of the n -form potentials of $D = 9$ maximal supergravity. The U-duality is $\text{SL}(2, \mathbb{R}) \times \mathbb{R}^+$. We denote with (w_1, w_2) the weights associated to $\mathbb{R}^+ \times \mathbb{R}^+$. The weight under T-duality is $w_1 - w_2$.

the 3-form (and 2-brane) is

$$A_{\mu\nu\rho} \rightarrow \text{D2} \quad (4.42)$$

and the 4-form (and 3-brane) is

$$A_{\mu_1 \dots \mu_4} \rightarrow \text{D3} \quad . \quad (4.43)$$

The fact that we have called F0 the brane associated to the field $g_\mu{}^9$, which is a reduced pp-wave, is straightforward from considering the T-dual IIB picture. For all the other cases the assignments are straightforward from the reduction of eq. (4.39).

There are a doublet and a singlet of 1-forms, and what eq. (4.40) shows is that the RR-form always belongs to the doublet. Besides, no matter how we choose the RR-1-form within the doublet, eq. (4.41) reveals that the RR-2-form must correspond to the same component.¹¹ A similar analysis can be performed for the higher rank branes. The decomposition of the 1-forms in terms of RR and F fields allows us to identify the q and \tilde{q} charges as

$$C_1 = \tilde{q}^\alpha A_{1,\alpha} \quad B_1 = q^\alpha A_{1,\alpha} \quad B'_1 = q A_1 \quad . \quad (4.44)$$

¹¹In comparing the doublet of eq. (4.40) with the doublet of eq. (4.41) one should keep in mind that the $\text{SL}(2, \mathbb{R})$ indices are raised and lowered by means of the epsilon symbol.

This breaks $SL(2, \mathbb{R})$ to the subgroup $SO(1, 1)$, which is isomorphic to \mathbb{R}^+ . We denote with w_1 the charge associated with this \mathbb{R}^+ , while the charge associated to the original \mathbb{R}^+ is denoted with w_2 . The actual T-duality group is a linear combination of these two \mathbb{R}^+ -factors such that the weight w under the T-duality group is given by $w = w_1 - w_2$. We summarise the decompositions of all the fields in table 6.

By analysing the table, one can determine the charges for all the other fields. For the 2-forms $A_{2,\alpha}$ one has

$$(-1, 1) : \tilde{q}^\alpha q \quad (1, 1) : q^\alpha q \quad , \quad (4.45)$$

for the 3-form A_3 the charge is

$$(0, 1) : \epsilon_{\alpha\beta} \tilde{q}^\alpha q^\beta q \quad (4.46)$$

and for the 4-form A_4 is

$$(0, 2) : \epsilon_{\alpha\beta} \tilde{q}^\alpha q^\beta q^2 \quad . \quad (4.47)$$

The expression for the RR and Fundamental 2-forms is thus

$$C_2 = \tilde{q}^\alpha q A_{2,\alpha} \quad B_2 = q^\alpha q A_{2,\alpha} \quad , \quad (4.48)$$

and similarly for the higher rank forms.

Denoting with $n(\tilde{q})$, $n(q)$ and $n'(q)$ the number of times the charges \tilde{q}^α , q^α and q respectively occur in the decomposition of a given charge, one has the relations

$$n(q) - n(\tilde{q}) = w_1 \quad n'(q) = w_2 \quad . \quad (4.49)$$

Using this and the actual $SL(2, \mathbb{R})$ representation to which each field belongs, the reader can identify the charges corresponding to the higher rank fields. For instance, the 9-form in the quadruplet is $A_{9,\alpha\beta\gamma}$, and its $(3, 4)$ component is projected by

$$(3, 4) : (\epsilon_{\alpha\beta} \tilde{q}^\alpha q^\beta) q^\gamma q^\delta q^\epsilon q^4 \quad , \quad (4.50)$$

which is a RR field ($n(\tilde{q}) = 1$), while for the 9-form in the second doublet, $A'_{9,\alpha}$, the component $(-1, 2)$ is projected by

$$(-1, 2) : (\epsilon_{\alpha\beta} \tilde{q}^\alpha q^\beta)^3 \tilde{q}^\gamma q^2 \quad , \quad (4.51)$$

which has $n(\tilde{q}) = 4$. Finally, the relation between the charges w_1 and w_2 of an n -form and the dilaton scaling of the tension of the corresponding $(n - 1)$ -brane (if any) in the string frame is

$$\alpha = \frac{1}{2}(w_1 + w_2 - n) \quad , \quad (4.52)$$

which is in agreement with (4.33) using (4.49) and

$$n = n(\tilde{q}) + n(q) + n'(q) \quad . \quad (4.53)$$

For convenience, we now write down explicitly the gauge transformations of the C and B fields. The two Fundamental 1-forms B_1 and B'_1 transform as

$$\delta B_1 = d\Sigma_0 \quad \delta B'_1 = d\Sigma'_0 \quad , \quad (4.54)$$

and we denote their fields strengths as

$$H_2 = dB_1 \quad H'_2 = dB'_1 \quad . \quad (4.55)$$

The Fundamental 2-form B_2 transforms as

$$\delta B_2 = d\Sigma_1 - \frac{1}{2}(\Sigma'_0 H_2 + \Sigma_0 H'_2) \quad , \quad (4.56)$$

where the relative normalisation between the two terms in brackets has been chosen for convenience (one can always change it by a field redefinition of the form $B_2 \rightarrow B_2 + B_1 B'_1$). The gauge invariant field strength is

$$H_3 = dB_2 + \frac{1}{2}(B'_1 H_2 + B_1 H'_2) \quad (4.57)$$

and in the next section we will be needing the Bianchi identity

$$dH_3 = H'_2 H_2 \quad . \quad (4.58)$$

From table 6 one obtains that the gauge transformations of the C fields of even rank are

$$\delta C_{2n} = d\lambda_{2n-1} + H_3 \lambda_{2n-3} - H'_2 \lambda_{2n-2} \quad , \quad (4.59)$$

while the gauge transformations of the C fields of odd rank are

$$\delta C_{2n+1} = d\lambda_{2n} + H_3 \lambda_{2n-2} - H_2 \lambda_{2n-1} \quad . \quad (4.60)$$

D=8. We consider the 8-dimensional theory from the IIA perspective, and we thus reduce the fields and branes in (4.39). We first consider the 0-branes. Upon reduction we obtain the following six 0-branes:

$$\begin{array}{lll} g_\mu{}^\sharp \rightarrow \text{D0} & g_\mu{}^9 \rightarrow \text{F0} & g_\mu{}^8 \rightarrow \text{F0} \\ A_{\mu 89} \rightarrow \text{D0} & A_{\mu 8\sharp} \rightarrow \text{F0} & A_{\mu 9\sharp} \rightarrow \text{F0} \quad . \end{array} \quad (4.61)$$

Using the $\text{SL}(3, \mathbb{R})$ epsilon symbol $\epsilon^{89\sharp} = 1$ we can write the last line as

$$A_\mu{}^\sharp \rightarrow \text{D0} \quad A_\mu{}^9 \rightarrow \text{F0} \quad A_\mu{}^8 \rightarrow \text{F0} \quad . \quad (4.62)$$

We therefore end up with two triplets ($M = 8, 9, \sharp$) in the $\overline{\mathbf{3}}$ representation of $\text{SL}(3, \mathbb{R})$:

$$\begin{aligned} A_{\mu, M1} &= (A_\mu{}^8, A_\mu{}^9, A_\mu{}^\sharp) = (\text{F0}, \text{F0}, \text{D0}) , \\ A_{\mu, M2} &= (g_\mu{}^8, g_\mu{}^9, g_\mu{}^\sharp) = (\text{F0}, \text{F0}, \text{D0}) . \end{aligned} \quad (4.63)$$

Together, the two triplets transform as a doublet $A_{\mu, M\alpha}$ ($\alpha = 1, 2$), i.e. as a $(\overline{\mathbf{3}}, \mathbf{2})$ representation of the U-duality group.

Next, we consider the 1-branes. We have

$$A_{\mu\nu\sharp} \rightarrow \text{F1} , \quad A_{\mu\nu 9} \rightarrow \text{D1} , \quad A_{\mu\nu 8} \rightarrow \text{D1} \quad . \quad (4.64)$$

field	U repr	RR	F	Rest
1-form	$(\bar{\mathbf{3}}, \mathbf{2})$	$(\mathbf{1}, \mathbf{2})_{-2}$	$(\mathbf{2}, \mathbf{2})_1$	
2-form	$(\mathbf{3}, \mathbf{1})$	$(\mathbf{2}, \mathbf{1})_{-1}$	$(\mathbf{1}, \mathbf{1})_2$	
3-form	$(\mathbf{1}, \mathbf{2})$	$(\mathbf{1}, \mathbf{2})_0$		
4-form	$(\bar{\mathbf{3}}, \mathbf{1})$	$(\mathbf{2}, \mathbf{1})_1$		$(\mathbf{1}, \mathbf{1})_{-2}$
5-form	$(\mathbf{3}, \mathbf{2})$	$(\mathbf{1}, \mathbf{2})_2$		$(\mathbf{2}, \mathbf{2})_{-1}$
6-form	$(\mathbf{8}, \mathbf{1})$	$(\mathbf{2}, \mathbf{1})_3$		$(\mathbf{2}, \mathbf{1})_{-3} + (\mathbf{1}, \mathbf{1})_0 + (\mathbf{3}, \mathbf{1})_0$
	$(\mathbf{1}, \mathbf{3})$			$(\mathbf{1}, \mathbf{3})_0$
7-form	$(\mathbf{6}, \mathbf{2})$	$(\mathbf{1}, \mathbf{2})_4$		$(\mathbf{2}, \mathbf{2})_1 + (\mathbf{3}, \mathbf{2})_{-2}$
	$(\bar{\mathbf{3}}, \mathbf{2})$			$(\mathbf{2}, \mathbf{2})_1 + (\mathbf{1}, \mathbf{2})_{-2}$
8-form	$(\mathbf{15}, \mathbf{1})$	$(\mathbf{2}, \mathbf{1})_5$		$(\mathbf{4}, \mathbf{1})_{-1} + (\mathbf{2}, \mathbf{1})_{-1} + (\mathbf{3}, \mathbf{1})_{-4} + (\mathbf{3}, \mathbf{1})_2 + (\mathbf{1}, \mathbf{1})_2$
	$(\mathbf{3}, \mathbf{3})$			$(\mathbf{2}, \mathbf{3})_{-1} + (\mathbf{1}, \mathbf{3})_2$
	$2 \times (\mathbf{3}, \mathbf{1})$			$2 \times [(\mathbf{2}, \mathbf{1})_{-1} + (\mathbf{1}, \mathbf{1})_2]$

Table 7. The decomposition of the n -form potentials of $D = 8$ maximal supergravity. The U-duality symmetry is $\text{SL}(3, \mathbb{R}) \times \text{SL}(2, \mathbb{R})$ and the T-duality is $\text{SL}(2, \mathbb{R}) \times \text{SL}(2, \mathbb{R})$, while the subscript denotes the \mathbb{R}^{+-} charge w .

This forms a single triplet in the $(\mathbf{3}, \mathbf{1})$ representation of the U-duality group:

$$A_{\mu\nu}^M = (A_{\mu\nu 8}, A_{\mu\nu 9}, A_{\mu\nu \sharp}) = (\text{D1}, \text{D1}, \text{F1}) . \quad (4.65)$$

Finally, we consider the 2-branes. These are

$$A_{\mu\nu\rho} \rightarrow \text{D2}, \quad A_{\mu\nu\rho 89\sharp} \rightarrow \text{D2}. \quad (4.66)$$

The two fields are $\text{SL}(3, \mathbb{R})$ singlets, which form an $\text{SL}(2, \mathbb{R})$ doublet $A_{3,\alpha}$.

By looking at equations (4.63), (4.65) and (4.66) one can see that there is an $\text{SL}(2, \mathbb{R})$ inside $\text{SL}(3, \mathbb{R})$ which leaves the D0-branes and the D2-branes invariant and transforms covariantly the D1-branes (in our choice this is the $\text{SL}(2, \mathbb{R})$ that rotates the first two components of the triplet). It is also evident that the D0-branes and the D2-branes transform covariantly with respect to the other $\text{SL}(2, \mathbb{R})$, while the D1-branes are invariant. The T-duality is thus $\text{SL}(2, \mathbb{R}) \times \text{SL}(2, \mathbb{R})$. In table 7 we give the decomposition of all the fields in terms of the T-duality group. We also give in the table the corresponding \mathbb{R}^{+-} charge as a subscript. The conventions for this charge in this case and in all the lower dimensional ones are taken from [31].

The table allows us to identify the q charges. In particular, from eq. (4.63) we see that the D0-branes are a doublet of the U-duality $\text{SL}(2, \mathbb{R})$, and we thus write

$$C_{1,\alpha} = \tilde{q}^M A_{1,M\alpha}, \quad B_{1,\alpha\dot{\alpha}} = q_{\dot{\alpha}}^M A_{1,M\alpha}, \quad (4.67)$$

where $\dot{\alpha}$ denotes the doublet of the $SL(2, \mathbb{R})$ inside $SL(3, \mathbb{R})$. All other charges projecting the higher rank forms can be uniquely expressed as products of these basic charges following the general analysis at the beginning of this section. Denoting with $n(\tilde{q})$ and $n(q)$ the number of times the charges \tilde{q}^M and $q_{\dot{\alpha}}^M$ occur in the decomposition of a given field, the \mathbb{R}^+ -charges of the n -form fields ($n = n(\tilde{q}) + n(q)$) are related to these numbers by

$$w = -2n(\tilde{q}) + n(q) \quad . \quad (4.68)$$

In case the form is associated to a brane, the dilaton scaling of the tension of a $D-(n-1)$ -brane is given in terms w as

$$g_s^\alpha, \quad \alpha = -\frac{1}{3}(n - w) \quad . \quad (4.69)$$

It is instructive to identify all the q charges for the higher rank fields applying to the eight-dimensional case the general analysis at the beginning of this section. For the two forms A_2^M eq. (4.22) becomes

$$(\mathbf{2}, \mathbf{1})_{-1} : \tilde{q}^N q_{\dot{\alpha}}^P \epsilon_{MNP} \quad (\mathbf{1}, \mathbf{1})_2 : q_{\dot{\alpha}}^N q_{\dot{\beta}}^P \epsilon^{\dot{\alpha}\dot{\beta}} \epsilon_{MNP} \quad , \quad (4.70)$$

so that the RR and Fundamental 2-forms are given by

$$C_{2,\dot{\alpha}} = \tilde{q}^N q_{\dot{\alpha}}^P \epsilon_{MNP} A_2^M \quad B_2 = q_{\dot{\alpha}}^N q_{\dot{\beta}}^P \epsilon^{\dot{\alpha}\dot{\beta}} \epsilon_{MNP} A_2^M \quad . \quad (4.71)$$

For the 3-forms $A_{3,\alpha}$ eq. (4.28) gives

$$(\mathbf{1}, \mathbf{2})_0 : \tilde{q}^M q_{\dot{\alpha}}^N q_{\dot{\beta}}^P \epsilon^{\dot{\alpha}\dot{\beta}} \epsilon_{MNP} \quad . \quad (4.72)$$

One can actually determine the charge associated to all the fields in table 7, not only the RR fields. For instance, for the 5-forms $A_5^M{}_\alpha$, one gets the charge projecting on the RR field, which is the $(\mathbf{1}, \mathbf{2})_2$,

$$(\mathbf{1}, \mathbf{2})_2 : \tilde{q}^N q_{\dot{\alpha}}^P q_{\dot{\beta}}^Q q_{\dot{\gamma}}^R q_{\dot{\delta}}^S \epsilon^{\dot{\alpha}\dot{\beta}} \epsilon^{\dot{\gamma}\dot{\delta}} \epsilon_{NMQR} \epsilon_{PRS} \quad , \quad (4.73)$$

which has $n(\tilde{q}) = 1$ and $n(q) = 4$, and the charge projecting on the $(\mathbf{2}, \mathbf{2})_{-1}$,

$$(\mathbf{2}, \mathbf{2})_{-1} : \tilde{q}^N \tilde{q}^P q_{\dot{\beta}}^Q q_{\dot{\gamma}}^R q_{\dot{\delta}}^S \epsilon^{\dot{\gamma}\dot{\delta}} \epsilon_{NMQR} \epsilon_{PRS} \quad , \quad (4.74)$$

which has $n(\tilde{q}) = 2$ and $n(q) = 3$. One can show that any other structure made of q 's and \tilde{q} 's with $n(q) + n(\tilde{q}) = 5$ projecting the 5-form vanishes identically.

As an example, we now wish to consider the non-trivial conjugacy classes to which the D5-branes belong. For this purpose, it is much easier to consider the truncation to the even form sector, that is all the even-form fields whose gauge transformations are generated by the gauge transformations of the 2-form. In that case the basic charges are given by

$$\tilde{Q}_{M\dot{\alpha}}, \quad Q_M \quad . \quad (4.75)$$

These basic charges define the RR and Fundamental 2-forms as

$$C_{2,\dot{\alpha}} = \tilde{Q}_{M\dot{\alpha}} A_2^M, \quad B_2 = Q_M A_2^M \quad . \quad (4.76)$$

The charges of the even-form fields that survive the truncation can be expressed as products of these basic charges. In particular, for the 4-form $A_{4,M}$ one gets

$$(\mathbf{2}, \mathbf{1})_1 : \tilde{Q}_{N\dot{\alpha}} Q_P \epsilon^{MNP} \quad (\mathbf{1}, \mathbf{1})_{-2} : \tilde{Q}_{N\dot{\alpha}} \tilde{Q}_{P\dot{\beta}} \epsilon^{\dot{\alpha}\dot{\beta}} \epsilon^{MNP} \quad , \quad (4.77)$$

while for the 6-form $A_{6,P}{}^Q$ one gets¹²

$$\begin{aligned} (\mathbf{2}, \mathbf{1})_3 : \tilde{Q}_{M\dot{\alpha}} Q_N Q_Q \epsilon^{MNP} & \quad (\mathbf{2}, \mathbf{1})_{-3} : \tilde{Q}_{M\dot{\alpha}} \tilde{Q}_{N\dot{\beta}} \tilde{Q}_{Q\dot{\gamma}} \epsilon^{\dot{\beta}\dot{\gamma}} \epsilon^{MNP} \\ (\mathbf{3}, \mathbf{1})_0 : \tilde{Q}_{Q(\dot{\alpha}} \tilde{Q}_{M\dot{\beta})} Q_N \epsilon^{MNP} & \quad (\mathbf{1}, \mathbf{1})_0 : \tilde{Q}_{M\dot{\alpha}} \tilde{Q}_{N\dot{\beta}} Q_Q \epsilon^{\dot{\alpha}\dot{\beta}} \epsilon^{MNP} \end{aligned} \quad (4.78)$$

The \mathbb{R}^+ - charges are related to $n(\tilde{Q})$ and $n(Q)$ as

$$w = 2n(Q) - n(\tilde{Q}) \quad . \quad (4.79)$$

Similar to the D7-branes in Type IIB string theory, we may now ask which of the eight 5-branes can be reached by an $\text{SL}(3, \mathbb{R})$ rotation of the two D5-branes, which correspond to the $(\mathbf{2}, \mathbf{1})_3$ fields. Consider the three independent three-vectors

$$\tilde{Q}_{M1}, \quad \tilde{Q}_{M2}, \quad Q_M \quad . \quad (4.80)$$

We observe that in the expression for the charge of the D-brane two of the three vectors are the same. This means that by an $\text{SL}(3, \mathbb{R})$ rotation one can reach only those branes whose expressions for the charge also contains two vectors that are the same. This applies to the two $(\mathbf{2}, \mathbf{1})_{-3}$ branes and to two of the three $(\mathbf{3}, \mathbf{1})_0$ branes. This implies that the D5-branes describe a non-linear sextuplet embedded into the octuplet. The two remaining branes have charges such that the three Q 's are all different. One may verify that for those branes one cannot write down a gauge-invariant WZ-term.

The same conclusion can be reached considering the 0-brane charges q and \tilde{q} . To summarise, the q 's and \tilde{q} 's select covariantly an $\text{SL}(2, \mathbb{R})$ inside $\text{SL}(3, \mathbb{R})$, and rotating these charges under $\text{SL}(3, \mathbb{R})$ corresponds to choosing a different embedding. All the components of a representation of $\text{SL}(3, \mathbb{R})$ that can be reached this way form a conjugacy class. This applies to any dimension.

D=7. We now consider the seven-dimensional case. From eq. (4.39) one finds that the 1-form fields associated to the D0-branes are

$$g_\mu{}^\sharp \quad A_{\mu 89} \quad A_{\mu 97} \quad A_{\mu 87} \quad . \quad (4.81)$$

There is a manifest $\text{SL}(4, \mathbb{R})$ symmetry associated to the torus, and using the corresponding epsilon symbol these fields can be written as

$$A_{\mu, 987} \quad A_\mu{}^\sharp{}^7 \quad A_\mu{}^\sharp{}^8 \quad A_\mu{}^\sharp{}^9 \quad . \quad (4.82)$$

¹²This 6-form is the $(\mathbf{8}, \mathbf{1})$. The other 6-form, the $(\mathbf{1}, \mathbf{3})$, disappears in the even-form truncation because its gauge transformations do not talk to the gauge transformations of the lower rank even forms.

field	U repr	RR	F	Rest
1-form	$\overline{\mathbf{10}}$	$\overline{\mathbf{4}}_{-3}$	$\mathbf{6}_2$	
2-form	$\mathbf{5}$	$\mathbf{4}_{-1}$	$\mathbf{1}_4$	
3-form	$\overline{\mathbf{5}}$	$\overline{\mathbf{4}}_1$		$\mathbf{1}_{-4}$
4-form	$\mathbf{10}$	$\mathbf{4}_3$		$\mathbf{6}_{-2}$
5-form	$\mathbf{24}$	$\overline{\mathbf{4}}_5$		$\mathbf{15}_0 + \mathbf{4}_{-5} + \mathbf{1}_0$
6-form	$\overline{\mathbf{40}}$	$\mathbf{4}_7$		$\overline{\mathbf{20}}_{-3} + \mathbf{10}_2 + \mathbf{6}_2$
	$\overline{\mathbf{15}}$			$\overline{\mathbf{10}}_2 + \overline{\mathbf{4}}_{-3} + \mathbf{1}_{-8}$
7-form	$\mathbf{70}$	$\overline{\mathbf{4}}_9$		$\mathbf{36}_{-1} + \mathbf{15}_4 + \mathbf{10}_{-6} + \mathbf{4}_{-1} + \mathbf{1}_4$
	$\mathbf{45}$			$\mathbf{20}_{-1} + \mathbf{15}_4 + \mathbf{6}_{-6} + \mathbf{4}_{-1}$
	$\mathbf{5}$			$\mathbf{4}_{-1} + \mathbf{1}_4$

Table 8. The decomposition of the n -form potentials of $D = 7$ maximal supergravity. The U-duality group is $\text{SL}(5, \mathbb{R})$ and the T-duality group is $\text{SL}(4, \mathbb{R})$. We denote as a subscript the \mathbb{R}^{+-} charge (notation from [31]).

Using the $\text{SL}(5, \mathbb{R})$ symmetry enhancement, and denoting with $6'$ the extra index of $\text{SL}(5, \mathbb{R})$, one can now use the corresponding epsilon symbol on the first vector to obtain

$$A_\mu{}^{\sharp 6'} \quad A_\mu{}^{\sharp 7} \quad A_\mu{}^{\sharp 8} \quad A_\mu{}^{\sharp 9} \quad . \quad (4.83)$$

Thus the D-branes correspond to the components of the vector $A_{\mu, MN}$ in the $\overline{\mathbf{10}}$ of $\text{SL}(5, \mathbb{R})$ with one index \sharp . The $\text{SL}(4, \mathbb{R})$ which transforms the other four indices is the T-duality. Therefore the D-branes belong to the $\overline{\mathbf{4}}$ of the T-duality group, while the remaining 1-forms have $\text{SL}(5, \mathbb{R})$ indices different from \sharp , which identifies the $\mathbf{6}$ of $\text{SL}(4, \mathbb{R})$. The corresponding branes are Fundamental. In order to realise this covariantly, we therefore introduce the charges \tilde{q}_a^{MN} and q_{ab}^{MN} that identify the RR and Fundamental 1-forms:

$$C_{1,a} = \tilde{q}_a^{MN} A_{1,MN} \quad B_{1,ab} = q_{ab}^{MN} A_{1,MN} \quad . \quad (4.84)$$

The 2-forms that result from the reduction of (4.39) are

$$A_{\mu\nu\sharp} \quad A_{\mu\nu 9} \quad A_{\mu\nu 8} \quad A_{\mu\nu 7} \quad A_{\mu\nu 789\sharp} \quad , \quad (4.85)$$

and the last component can be rewritten as $A_{\mu\nu 6'}$, which makes the $\mathbf{5}$ of $\text{SL}(5, \mathbb{R})$. From (4.39) it follows that the first component $A_{\mu\nu\sharp}$ corresponds to the Fundamental string, while the other components are associated to D1-branes. This is in agreement with eq. (4.22), which in this seven-dimensional case says that the 2-forms A_2^M are projected on the RR 2-forms in the $\mathbf{4}$ of $\text{SL}(4, \mathbb{R})$ by

$$C_2^a = \tilde{q}_b^{NP} q_{cd}^{QR} \epsilon^{abcd} \epsilon_{MNPQR} A_2^M \quad (4.86)$$

field	U repr	RR	F	Rest
1-form	16	$(\mathbf{8}_S)_{-1}$	$(\mathbf{8}_C)_1$	
2-form	10	$(\mathbf{8}_V)_0$	$\mathbf{1}_2$	$\mathbf{1}_{-2}$
3-form	$\overline{\mathbf{16}}$	$(\mathbf{8}_S)_1$		$(\mathbf{8}_C)_{-1}$
4-form	45	$(\mathbf{8}_V)_2$		$(\mathbf{8}_V)_{-2} + \mathbf{28}_0 + \mathbf{1}_0$
5-form	144	$(\mathbf{8}_S)_3$		$(\mathbf{8}_C)_1 + (\mathbf{8}_V)_{-1} + (\mathbf{8}_C)_{-3} + (\mathbf{56}_V)_{-1} + (\mathbf{56}_C)_1$
6-form	320	$(\mathbf{8}_V)_4$		$(\mathbf{8}_V)_{-4} + 2 \times (\mathbf{8}_V)_0 + (\mathbf{35}_V)_2 + (\mathbf{35}_V)_{-2} + (\mathbf{160}_V)_0$ $+ \mathbf{28}_2 + \mathbf{28}_{-2} + \mathbf{1}_2 + \mathbf{1}_{-2}$
	$\overline{\mathbf{126}}$			$(\mathbf{35}_S)_2 + (\mathbf{35}_C)_{-2} + (\mathbf{56}_V)_0$
	10			$(\mathbf{8}_V)_0 + \mathbf{1}_2 + \mathbf{1}_{-2}$

Table 9. The decomposition of the n -form potentials of $D = 6$ maximal supergravity. The U-duality is $\text{SO}(5, 5)$, while the T-duality is $\text{SO}(4, 4)$.

and on the Fundamental 2-form by

$$B_2 = q_{ab}^{NP} q_{cd}^{QR} \epsilon^{abcd} \epsilon_{MNPQR} A_2^M \quad . \quad (4.87)$$

The decomposition of all the fields under T-duality is given in table 8. The relation between the \mathbb{R}^{+-} charge w and the q 's is

$$w = -3n(\tilde{q}) + 2n(q) \quad . \quad (4.88)$$

From this one can determine all the \mathbb{R}^{+-} charges of the various T-duality representations given in table 8.

D=6. In six dimensions the U-duality group is $\text{SO}(5, 5)$ which is decomposed under the T-duality group $\text{SO}(4, 4)$. This is given in table 9. We follow the group theory conventions of [31], and therefore the RR 2-forms belong to the $\mathbf{8}_V$ of $\text{SO}(4, 4)$. This is not in contradiction with the general case, in which all RR fields are in the spinor representations of the T-duality group, because of triality of $\text{SO}(4, 4)$.

As usual, we denote with w the \mathbb{R}^{+-} charge. For the cases in which the n -form can be associated to a brane, the corresponding tension scales in the string frame as

$$\alpha = \frac{1}{2}(w - n) \quad . \quad (4.89)$$

Only one of the two singlets that arise in the decomposition of the 2-forms is a Fundamental string. The other singlet corresponds to a string scaling like g_s^{-2} , which is the magnetic dual of the Fundamental string.

D=5. In five dimensions the 1-forms, which belong to the **27** of E_6 , decompose into $\mathbf{16} + \mathbf{10} + \mathbf{1}$ under T-duality. The decomposition of all the fields is given in table 10,

field	U repr	RR	F	Rest
1-form	27	16 ₁	10 ₋₂	1 ₄
2-form	27	16 ₋₁	1 ₋₄	10 ₂
3-form	78	16 ₋₃		16 ₃ + 45 ₀ + 1 ₀
4-form	351	16 ₋₅		16 ₁ + 45 ₄ + 120 ₋₂ + 144 ₁ + 10 ₋₂
5-form	1728	16 ₋₇		1 ₋₄ + 10 ₂ + 2 × 16 ₋₁ + 45 ₋₄ + 120 ₂ + 126 ₂ + 144 ₋₁ + 144 ₅ + 210 ₋₄ + 320 ₂ + 560 ₋₁
	27			16 ₋₁ + 10 ₂ + 1 ₋₄

Table 10. The decomposition of the n -form potentials of $D = 5$ maximal supergravity. The U-duality is E_6 and the T-duality is $SO(5,5)$.

where the subscript denotes the \mathbb{R}^+ -charge w . For the cases in which an $(n - 1)$ -brane is associated to the n -form, the corresponding tension scales in the string frame as

$$\alpha = -\frac{1}{3}(w + 2n) \quad . \quad (4.90)$$

This shows that the 1-form singlet is not a Fundamental particle. This is the highest dimension in which the 1-forms are not completely decomposed into RR and Fundamental fields.

D=4. We first consider the 1-forms, see table 11. The RR 1-forms occur as a 32-dimensional spinor representation of the T-duality group $SO(6,6)$, as expected. There are 24 remaining 1-forms, half of them correspond to Fundamental 0-branes, the other half correspond to Solitonic 0-branes with dilaton coupling $1/g_s^2$. To pick out the Fundamental fields one must therefore undo the symmetry enhancement by making the decomposition (4.5). Under this decomposition the 1-forms branch as

$$(\mathbf{32}, \mathbf{1}) \rightarrow \mathbf{32}_0, \quad (\mathbf{12}, \mathbf{2}) \rightarrow \mathbf{12}_{-1} + \mathbf{12}_1, \quad (4.91)$$

where the sub-index indicates the weight w under \mathbb{R}^+ . One next uses the rule that the brane tension corresponding to an n -form scales as

$$g_s^\alpha, \quad \alpha = -(n + w). \quad (4.92)$$

This confirms that the $\mathbf{32}_0$ charges describe 32 D0-branes. We furthermore deduce that the $\mathbf{12}_{-1}$ charges describe 12 Fundamental 0-branes and that the $\mathbf{12}_1$ charges describe 12 Solitonic 0-branes.

Like in $D = 5$ dimensions, the 1-forms are decomposed not only in RR and Fundamental fields but also in Solitonic fields D . We therefore introduce the three basic charges $(\tilde{q}_a^M, q_A^M, q_A'^M)$ and decompose the 1-forms as follows:

$$C_{1,a} = \tilde{q}_a^M A_{1,M}, \quad B_{1,A} = q_A^M A_{1,M}, \quad D_{1,A} = q_A'^M A_{1,M}. \quad (4.93)$$

In terms of these charges the weight w of an n -form is given by

$$w = -n(q) + n(q'). \quad (4.94)$$

We next consider the 2-forms $A_{2,\alpha}$. Under the decomposition (4.5) the 2-forms in the RR column branch according to

$$(\mathbf{32}', \mathbf{2}) \rightarrow \mathbf{32}'_{-1} + \mathbf{32}'_1 \quad (4.95)$$

This shows that the $\mathbf{32}'_{-1}$ 2-forms describe D1-branes but that the $\mathbf{32}'_1$ 2-forms describe exotic objects with $1/g_s^3$ dilaton coupling. Their charges are given by

$$\mathbf{32}'_{-1} : \quad \tilde{q}_a^M q_A^N (\Gamma^A)_{\dot{a}}^a D_{MN}^\alpha, \quad \mathbf{32}'_1 : \quad \tilde{q}_a^M q_A^N (\Gamma^A)_{\dot{a}}^a D_{MN}^\alpha. \quad (4.96)$$

Similarly, the 2-forms in the F column decompose according to

$$(\mathbf{1}, \mathbf{3}) \rightarrow \mathbf{1}_{-2} + \mathbf{1}_0 + \mathbf{1}_2. \quad (4.97)$$

They describe a Fundamental string, a Solitonic string and an exotic object with $1/g_s^4$ coupling, respectively. Their charges are given by

$$\mathbf{1}_{-2} : \quad q_A^M q_B^N \eta^{AB} D_{MN}^\alpha, \quad \mathbf{1}_0 : \quad q_A^M q_B^N \eta^{AB} D_{MN}^\alpha, \quad \mathbf{1}_2 : \quad q_A^M q_B^N \eta^{AB} D_{MN}^\alpha. \quad (4.98)$$

Finally, the remaining 2-forms in the Rest column decompose according to

$$(\mathbf{66}, \mathbf{1}) \rightarrow \mathbf{66}_0. \quad (4.99)$$

They correspond to Solitonic strings with charges given by

$$\mathbf{66}_0 : \quad q_{[A}^M q_{B]}^N D_{MN}^\alpha. \quad (4.100)$$

Similarly, one can discuss the higher n -forms.

5 D-brane WZ terms

In this section we will derive the main result of this paper, i.e. the expression (1.14) for the general D-brane WZ term in $3 \leq D \leq 10$ dimensions. Our starting point is the U-duality covariant expressions for the RR C fields in terms of the covariant A fields, derived in the previous section. These RR fields have the important property that under gauge transformations they transform only into themselves, see eq. (4.14). They form the building blocks for our construction of a gauge-invariant WZ term.

It is well-known that in ten dimensions it is not possible to construct a gauge-invariant WZ term using the RR fields C alone. The reason is that the C fields not only transform into a total derivative but also into a term containing the Fundamental curvature H_3 which has to be cancelled. Trying something of the form $e^{X_2} C$ with $dX_2 = H_3$ would solve the problem. Taking $X_2 = B_2$ is not allowed since X_2 has to be gauge-invariant by itself. This is the reason that we need to introduce a BI vector such that B_2 can be

field	U repr	RR	F	Rest
1-form	56	(32, 1)	(12, 2)	
2-form	133	(32', 2)	(1, 3)	(66, 1)
3-form	912	(32, 3)		(12, 2) + (352, 1) + (220, 2)
4-form	8645	(32', 4)		(2079, 1) + (1728, 2) + (495, 3) + (462, 1) + (352, 2) + (66, 3) + (66, 1) + (32', 2) + (1, 3)
	133			(32', 2) + (1, 3) + (66, 1)

Table 11. The decomposition of the n -form potentials of $D = 4$ maximal supergravity. The U-duality is E_7 and the T-duality is $SO(6, 6)$ with symmetry enhancement to $SO(6, 6) \times SL(2, \mathbb{R})$. The n -forms in the RR and F columns do not only contain the RR and Fundamental fields but also fields with different dilaton couplings, see the text.

interpreted as a term inside the gauge-invariant curvature of the BI vector. We therefore take $X_2 = \mathcal{F}_2 = dV_1 + B_2$.

In $D < 10$ dimensions a similar reasoning works except that the C fields now not only transform to H_3 but also to $H_{2,A}$, the curvature of the Fundamental 1-forms $B_{1,A}$. This suggests that we need to introduce not only a BI vector V_1 but also $2(10 - D)$ worldvolume scalars $V_{0,A}$ with corresponding gauge invariant field strengths given by

$$\mathcal{F}_{1,A} = dV_{0,A} + B_{1,A}. \quad (5.1)$$

This is gauge-invariant provided that the worldvolume scalars transforms as

$$\delta V_{0,A} = -\Sigma_{0,A}. \quad (5.2)$$

Furthermore, we need to adapt the definition of \mathcal{F}_2 since B_2 also transforms under $\Sigma_{0,A}$, see eq. (4.12). The following expression is gauge-invariant

$$\mathcal{F}_2 = dV_1 + B_2 - V_{0,A} H_{2,B} \eta^{AB}, \quad (5.3)$$

provided that V_1 transforms under gauge transformations as

$$\delta V_1 = -\Sigma_1. \quad (5.4)$$

It is instructive to compare the above with the expected number of bosonic worldvolume degrees of freedom. In general a Dp -brane has 8 bosonic worldvolume degrees of freedom. For instance, in ten dimensions a Dp -brane has $(p - 1)$ d.o.f. represented by the BI vector and $(10 - p - 1)$ d.o.f. represented by the worldvolume scalars, after fixing the $(p + 1)$ worldvolume reparametrisations, i.e.

$$(p - 1) + (10 - p - 1) = 8. \quad (5.5)$$

In $D < 10$ dimensions we not only have the $(p - 1)$ d.o.f. represented by the BI vector and the $D - p - 1$ d.o.f. of the embedding scalars but also the $(10 - D)$ d.o.f. represented by the

wrapping of the Fundamental string around each of the $(10 - D)$ compactified dimensions. This leads to the same total number of d.o.f. as in $D = 10$:

$$(p - 1) + (D - p - 1) + (10 - D) = 8. \quad (5.6)$$

We find that the WZ term contains twice as much extra scalars than expected, i.e. $2(10 - D)$ instead of $(10 - D)$. We will comment about this in the conclusions section.

Given these ingredients, we can now write a compact expression for the gauge-invariant WZ term for any D-brane in any dimension. The result is given in (1.14) which we repeat here:

$$\mathcal{L}_{\text{WZ}}(D < 10) = e^{\mathcal{F}_2} e^{\mathcal{F}_{1,A} \Gamma^A} C \quad . \quad (5.7)$$

Like in the previous section we denote with C the sum of all the RR potentials. The $D = 9$ case is a bit special in the sense that the combination $\mathcal{F}_{1,A} \Gamma^A$ only contains the self-dual (anti-self-dual) part of $\mathcal{F}_{1,A}$ when projected on the even (odd) form sector. These 1-forms contain the $(1, 0)$ and $(0, 1)$ 1-forms in table 6. It is also easier to explicitly write out the Γ^A matrices for this case. We therefore treat this case separately below, see (5.15) for an expression for the WZ term in that case.

To proof that the WZ term (5.7) is gauge-invariant we need the Bianchi identities of the worldvolume curvatures:

$$\begin{aligned} d\mathcal{F}_{1,A} &= H_{2,A} \, , \\ d\mathcal{F}_2 &= H_3 - \mathcal{F}_{1,A} H_{2,B} \eta^{AB} \quad . \end{aligned} \quad (5.8)$$

The proof of gauge-invariance is now remarkably simple. Given that the \mathcal{F} 's are gauge invariant, we have

$$\delta \mathcal{L}_{\text{WZ}}(D < 10) = e^{\mathcal{F}_2} e^{\mathcal{F}_{1,A} \Gamma^A} (d\lambda + H_3 \lambda - H_{2,B} \Gamma^B \lambda) \quad . \quad (5.9)$$

The H_3 term cancels up to a total derivative precisely like in ten dimensions by using the second Bianchi identity of (5.8). We are now left with the following three terms, leaving out an overall $e^{\mathcal{F}_2}$ factor and the gauge parameter λ :

$$e^{\mathcal{F}_{1,A} \Gamma^A} \mathcal{F}_{1,B} H_{2,C} \eta^{BC} - d e^{-\mathcal{F}_{1,A} \Gamma^A} - e^{\mathcal{F}_{1,A} \Gamma^A} H_{2,B} \Gamma^B \quad . \quad (5.10)$$

The first term arises from the fact that we applied the second Bianchi identity of (5.8) when cancelling the H_3 term. The second term arises from partially differentiating the exterior derivative in (5.9) when it hits the $e^{\mathcal{F}_{1,A} \Gamma^A}$ term. Finally, the third term is just the last term of (5.9). To show that the three terms given in (5.10) cancel amongst each other it is convenient to first expand the exponential in the first and second term

$$e^{\mathcal{F}_{1,A} \Gamma^A} = \sum_{n=0}^{2(10-D)} \frac{1}{n!} \mathcal{F}_{1,A_1} \cdots \mathcal{F}_{1,A_n} \Gamma^{A_1 \cdots A_n} \quad (5.11)$$

and similarly expand the exponential in the second term. In the second term one next uses the first Bianchi identity of (5.8). Now all terms are linear in H_2 . After expanding

the exponentials the first and second term of (5.10) are written as a sum of completely anti-symmetric Gamma matrices. The third term becomes the sum of products of an anti-symmetric Gamma matrix with a single Γ_B matrix. Working out this product leads to two types of terms containing a single anti-symmetric Gamma matrix. In the first type H_2 is contracted with one of the indices of the anti-symmetric Gamma matrix. Such terms cancel against the second term of (5.10). In the second type H_2 is contracted with one of the \mathcal{F}_1 curvatures. These terms cancel against the first term of (5.10). This completes the proof of gauge invariance of the WZ term (5.7).

The form of the WZ term (5.7) clearly suggests that, like in $D = 10$, the RR scalars $C_{0,\dot{a}}$ can be included too in the expression although they are not needed for gauge-invariance. Like in IIB supergravity all RR scalars are axionic, i.e. the D -dimensional maximal supergravity theory is invariant under constant shifts of these scalars.

Given that in nine dimensions the T-duality group is abelian, we consider this case explicitly although it does not differ from the general analysis. We introduce the world-volume scalars V_0 and V'_0 , such that

$$\mathcal{F}_1 = dV_0 + B_1 \quad \mathcal{F}'_1 = dV'_0 + B'_1 \quad (5.12)$$

are gauge invariant. From (4.56) one also defines

$$\mathcal{F}_2 = dV_1 + B_2 - \frac{1}{2}(V'_0 H_2 + V_0 H'_2) \quad , \quad (5.13)$$

so that the Bianchi identities are

$$\begin{aligned} d\mathcal{F}_1 &= H_2 & d\mathcal{F}'_1 &= H'_2 \\ d\mathcal{F}_2 &= H_3 - \frac{1}{2}(\mathcal{F}'_1 H_2 + \mathcal{F}_1 H'_2) \quad . \end{aligned} \quad (5.14)$$

From the gauge transformations (4.59) and (4.60) one can then write a gauge invariant WZ term as the formal expression

$$\mathcal{L}_{\text{WZ}}(D=9) = e^{\mathcal{F}_2} e^{\mathcal{F}_1 + \mathcal{F}'_1} C \quad . \quad (5.15)$$

It is understood here that when we expand $e^{\mathcal{F}_1 + \mathcal{F}'_1}$ we only take \mathcal{F}_1 acting on even forms and \mathcal{F}'_1 acting on odd forms. That is

$$e^{\mathcal{F}_1 + \mathcal{F}'_1} C = \left(1 + \mathcal{F}_1 + \frac{1}{2}\mathcal{F}'_1 \mathcal{F}_1\right) C_{\text{even}} + \left(1 + \mathcal{F}'_1 + \frac{1}{2}\mathcal{F}_1 \mathcal{F}'_1\right) C_{\text{odd}} \quad . \quad (5.16)$$

Using this notation the proof of gauge-invariance is straightforward and will not be repeated here. The alternating occurrence of \mathcal{F}_1 and \mathcal{F}'_1 is easily explained if one considers the IIA and IIB origin of the WZ term. Indeed, an even form in nine dimensions is unwrapped from the IIB perspective and wrapped from the IIA perspective, and vice-versa for an odd form. Any time there is a wrapped coordinate, this corresponds to a world volume scalar appearing in the WZ term. In particular the scalar V_0 can be seen as arising from a wrapped world-volume vector V_1 in IIB, while the scalar V'_0 arises as a wrapped V_1 in IIA.

6 Conclusions

In this paper we have constructed gauge-invariant and U-duality covariant expressions for Wess-Zumino terms corresponding to general Dp -branes ($0 \leq p \leq D-1$) in arbitrary $3 \leq D \leq 10$ dimensions. We did this in two steps. First, we considered the target space background fields. In particular, we constructed expressions for the RR potentials in terms of the U-duality covariant fields. For this we introduced two types of charge vectors that project the 1-forms onto the RR and Fundamental 1-forms.¹³ We showed how, for $D < 10$, the charges of all Dp -branes with $p \geq 1$ could be expressed as products of these basic charges and we derived general expressions for these higher-dimensional charges. The cases $D = 3$ and $D = 4$ required special attention due to symmetry enhancements that take place in these dimensions. Since the extra symmetries put the D-branes together with other objects into the same multiplet it is natural, for the purposes of this paper, to undo these symmetry enhancements. We discussed the $D = 4$ case in quite some detail. We refrained from giving the formulae for the $D = 3$ case as well but we expect that it follows the same pattern we found in higher dimensions.

In a second step we considered the worldvolume fields needed for the construction of the WZ term. Since, for $D < 10$, the WZ term for general D-branes contains both even-form and odd-form potentials it is clear that one needs also even-form and odd-form worldvolume curvatures. We therefore introduced, besides the usual worldvolume 2-form curvature for the BI vector, additional worldvolume 1-form curvatures for the extra worldvolume scalars that correspond to the compactified dimensions. These scalars are on top of the usual embedding scalars. Using the expressions for the RR potentials and the worldvolume curvatures it was then a relatively straightforward task to construct a gauge-invariant and duality-covariant expression for the WZ term in $D < 10$ dimensions.

The fact that for $D < 10$ the charges of the higher-dimensional branes can be expressed in terms of products of the D0-brane and Fundamental 0-brane charges has important consequences for the non-trivial conjugacy classes to which these D-branes may belong. As an example, we analysed in detail the case of D5-branes in $D = 8$ dimensions. To simplify matters we did this for the truncated case of even forms only. Making use of the observation that the charge of the standard D5-brane contains the product of two charge vectors that are the same we showed that under a general U-duality transformation they form a non-linear six-plet embedded into an octo-plet of 5-branes. This is similar to the D7-branes in IIB string theory which form a non-linear doublet inside a triplet of 7-branes. Other examples of non-trivial conjugacy classes may be discussed similarly.

A noteworthy feature of the general WZ term is that it contains twice as many extra scalars as compactified dimensions. One set of scalars has a natural IIA origin, the other set has a natural IIB origin. Together they transform as a vector of the T-duality group $SO(10-D, 10-D)$. This doubling of compactified dimensions is typical for doubled geometry [32–35] but now applied to the worldvolume of the D-branes in a curved background. In the same way that in dimensions lower than ten the Wess-Zumino term of the $D = 10$

¹³Actually, for $D \leq 5$ we need extra basic charges to project onto Solitonic 1-forms as well.

fundamental string, $\mathcal{L}_{F1} = B_2$, gets modified by $\mathcal{F}_{1,A}$ to [32]

$$\mathcal{L}_{F1}(D < 10) = B_2 + \eta^{AB} \mathcal{F}_{1,A} B_{1,B} \quad , \quad (6.1)$$

which is invariant under the NS-NS gauge transformations (4.12), we have derived that in $D < 10$ dimensions the WZ term of the $D = 10$ D-branes, $\mathcal{L}_{Dp} = e^{\mathcal{F}_2} C$ gets modified by $\mathcal{F}_{1,A}$ to the expression (5.7). In the case of the fundamental string a correct counting of the worldvolume scalars is re-obtained by imposing a self-duality condition on the scalars [32]. The challenge will be to see what kind of condition on the worldvolume scalars must be imposed, in the case of D-branes, to obtain a correct counting of the worldvolume degrees of freedom. Given that such a condition can be imposed for the fundamental string, we expect this to be a solvable problem [41].

There are two natural extensions of our work. One extension is to consider the coupling of D-branes to maximal *gauged* supergravities. This requires the introduction of the so-called embedding tensor [36–38]. Like in the case of the coupling of the D2-brane to massive IIA supergravity, we expect this embedding tensor to occur as the coefficient of a worldvolume Chern-Simons term [6]. More information about these extra Chern-Simons terms can be obtained by considering generalized Scherk-Schwarz reductions of the axionic RR scalars since these reductions lead to Chern-Simons terms. We plan to come back to the relation between the D-brane WZ terms and the embedding tensor in the nearby future [39].

A second extension is to construct a kappa-symmetric version of the D-brane actions. This requires of course first to construct a U-duality covariant kinetic term for all worldvolume fields. This has already been done for the $D = 10$ D-branes, see [10], and a kappa-symmetric extension of the $D = 10$ D-brane actions has been constructed [40].

Finally, we observe that lower-dimensional supergravities allow for many potentials that correspond to exotic branes with unconventional dilaton couplings $1/g_s^n$ ($n = 3, 4, 5, \dots$). It is not clear what the status of all these exotic branes is in string theory, and if they exist et all, but if they do exist they might help in finding a geometrical description of the individual degrees of freedom of black holes and thereby explaining the entropy of these black holes [42].

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